ON RANDOM POLYNOMIALS SPANNED BY OPUC

by

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To my parents and siblings

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ABSTRACT

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We consider the behavior of zeros of random polynomials of the from

$$P_{n,m}(z) := \eta_0 \varphi_m^{(m)}(z) + \eta_1 \varphi_{m+1}^{(m)}(z) + \cdots + \eta_n \varphi_{n+m}^{(m)}(z)$$

as $n \to \infty$, where m is a non-negative integer (most of the work deal with the case $m = 0$), $\{\eta_n\}_{n=0}^{\infty}$ is a sequence of i.i.d. Gaussian random variables, and $\{\varphi_n(z)\}_{n=0}^{\infty}$ is a sequence of orthonormal polynomials on the unit circle $\mathbb{T}$ for some Borel measure μ on $\mathbb{T}$ with infinitely many points in its support. Most of the work is done by manipulating the density function for the expected number of zeros of a random polynomial, which we call the intensity function.

1. INTRODUCTION

Random polynomials is a relatively old subject with initial contributions by Bloch and Pólya, Littlewood and Offord, Erdös and Offord, Arnold, Kac, and many other authors and that have applications in several fields of physics, engineering and economics. An interested reader can find a well referenced early history of the subject in the books by Bharucha-Reid and Sambandham [1], and by Farahmand [2]. In this document, we are concerned with the behavior of zeros of random orthogonal polynmoials on the unit circle. We proceed by introducing some basic property of orthogonal polynomials on the unit circle, OPUC. After that, a brief literature review of directly related studies is given.

1.1 Basic Facts about OPUC

Definition 1.1.1. *The sequence* $\{\varphi_n(z)\}_{n=0}^{\infty}$ *is called a sequence of orthonormal polynomials on the unit circle* $\mathbb{T}$ *if it satisfies*

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_k(z)}\mathrm{d}\mu(z) = \delta_{nk} \tag{1.1}$$

for some Borel measure μ *on* $\mathbb{T}$ *with infinitely many points in its support,* δ_{nk} *here is the usual Kronecker symbol that is defined as follows* $\delta_{nk} = 1$ *when* $n = k$ *and* $\delta_{nk} = 0$ *otherwise.*

The polynomial $\varphi_n(z)$ can be written as $\varphi_n(z) = \kappa_n \Phi_n(z)$, where $\Phi_n(z)$ is the monic orthogonal polynomial. Therefore, $\kappa_n = \|\Phi_n(z)\|^{-1}$.

1.1.1 Szegő Recurrence

A widely used property of OPUC is the recurrence relations. This property is stated in [3, Theorem 1.5.2] and we will restate here for reference. Define the reversed polynomial $\Phi_n^*(z)$ to be the polynomial of degree n given by,

$$\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$$

Notice that this operation only reverses the order of coefficients and takes complex conjugates. It turns out that monic orthogonal polynomials satisfy recurrence relations, known as Szegő recurrence, of the form

$$\begin{cases} \Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z), \\ \Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z), \end{cases} \tag{1.2}$$

where the recurrence coefficients $\{\alpha_n\}$ belong to the unit disk $\mathbb{D}$ and are uniquely determined by the measure of orthogonality. Moreover,

$$\|\Phi_{n+1}\|^2 = \Big(1 - |\alpha_n|^2\Big)\|\Phi_n\| = \prod_{i=0}^{n} 1 - |\alpha_i|^2.$$

Introducing the notation $\rho_i := (1 - |\alpha_i|^2)^{1/2}$, it holds that

$$\kappa_n = \|\Phi_n(z)\|^{-1} = \prod_{i=0}^{n-1} \rho_i^{-1} = \prod_{i=0}^{n-1} \Big(1 - |\alpha_i|^2\Big)^{-1/2}. \tag{1.3}$$

Another widely used result about OPUC which can be derived using the recurrence relation is Christoffel-Darboux formula

$$\begin{aligned} K_n(z,w) = \sum_{i=0}^{n} \varphi_i(z)\overline{\varphi_i(w)} &= \frac{\varphi_{n+1}^*(z)\overline{\varphi_{n+1}^*(w)} - \varphi_{n+1}(z)\overline{\varphi_{n+1}(w)}}{1 - z\overline{w}} \\ &= \frac{\varphi_n^*(z)\overline{\varphi_n^*(w)} - z\overline{w}\varphi_n(z)\overline{\varphi_n(w)}}{1 - z\overline{w}} \end{aligned} \tag{1.4}$$

for any $n \geq 0$ and $z, w \in \mathbb{C}$ with $z\overline{w} \neq 1$. See [3, Theorem 2.2.7] for derivation.

1.2 Brief Literature Review of the Studies of Expected Number of Zeros of Random Polynomials

The expected number of zeros of random polynomials is a widely studied subject. In this section, we will try to cover some of the result that provides an insight into our works.

In [4], Kac considered random polynomials of the form

$$P_n(z) = \eta_0 + \eta_1 z + \cdots + \eta_n z^n, \tag{1.5}$$

where η_i are i.i.d. standard real Gaussian random variables. He has shown that $\mathbb{E}_n(\Omega)$, the expected number of zeros of $P_n(z)$ on a measurable set $\Omega \subset \mathbb{R}$, is equal to

$$\mathbb{E}_n(\Omega) = \frac{1}{\pi} \int_\Omega \frac{\sqrt{1 - h_{n+1}^2(x)}}{|1 - x^2|} \mathrm{d}x, \quad h_{n+1}(x) = \frac{(n+1)x^n(1-x^2)}{1 - x^{2n+2}}, \tag{1.6}$$

from which he proceeded with an estimate

$$\mathbb{E}_n(\mathbb{R}) = \frac{2 + o(1)}{\pi} \log(n+1) \quad \text{as} \quad n \to \infty. \tag{1.7}$$

He has also showed that

$$\mathbb{E}_n(\mathbb{R}) \leq \frac{2}{\pi} \log(n+1) + \frac{14}{\pi}, \quad n \geq 1.$$

This result has been improved by several authors. For example, Wilkins [5] improved the estimate and obtained the result

$$\mathbb{E}_n(\mathbb{R}) \leq \frac{2}{\pi} \log(n+1) + 1.116.$$

Later, it was shown by Wilkins [6], after some intermediate results cited in [6], that there exist constants A_p, $p \geq 0$, such that $\mathbb{E}_n(\mathbb{R})$ has an asymptotic expansion of the form

$$\mathbb{E}_n(\mathbb{R}) \sim \frac{2}{\pi} \log(n+1) + \sum_{p=0}^{\infty} A_p (n+1)^{-p}, \tag{1.8}$$

where

$$A_0 = \frac{2}{\pi} \left(\log 2 + \int_0^1 \frac{f(t)}{t} \mathrm{d}t + \int_1^\infty \frac{f(t) - 1}{t} \mathrm{d}t \right), \quad f(t) := \sqrt{1 - \left(\frac{2t}{e^t - e^{-t}} \right)^2}. \tag{1.9}$$

Many subsequent results on random polynomials were concerned with relaxing the conditions on random coefficients, see, for example, [7–9], or the behavior of the counting measures of zeros of random polynomials spanned by various deterministic bases with random coefficients that are not necessarily Gaussian nor i.i.d. as in [10–27]. In the case of Kac polynomials these normalized counting measures almost surely converge to the arclength distribution on the unit circle ($\log n$ real zeros are clearly negligible when normalized by $1/n$). Our primary interest lies in studying the expected number of real zeros when the basis is a family of orthogonal polynomials in the spirit of [28–31]. More precisely, Edelman and Kostlan [32] considered random functions of the form

$$P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_n f_n(z), \tag{1.10}$$

where η_i are certain real random variables and $f_i(z)$ are arbitrary functions on the complex plane that are real on the real line. Using beautiful and simple geometrical argument they have shown[1] that if $\eta_0, \ldots, \eta_n$ are elements of a multivariate real normal distribution with mean zero and covariance matrix C and the functions $f_i(x)$ are differentiable on the real line, then

$$\mathbb{E}_n(\Omega) = \int_\Omega \rho_n(x)\mathrm{d}x, \quad \rho_n(x) = \frac{1}{\pi}\frac{\partial^2}{\partial s \partial t}\log\Big(v(s)^\mathsf{T} C v(t)\Big)\Big|_{t=s=x},$$

where $v(x) = \Big(f_0(x), \ldots, f_n(x)\Big)^\mathsf{T}$. If random variables η_i in (1.10) are again i.i.d. standard real Gaussians, then the above expression for $\rho_n(x)$ specializes to

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{K_{n+1}(x,x)K_{n+1}^{(1,1)}(x,x) - K_{n+1}^{(1,0)}(x,x)^2}}{K_{n+1}(x,x)} \tag{1.11}$$

(this formula was also independently rederived in [21, Proposition 1.1] and [33, Theorem 1.2]), where

$$\begin{cases} K_{n+1}(z,w) & := & \sum_{i=0}^n f_i(z)\overline{f_i(w)}, \\ K_{n+1}^{(1,0)}(z,w) & := & \sum_{i=0}^n f_i'(z)\overline{f_i(w)}, \\ K_{n+1}^{(1,1)}(z,w) & := & \sum_{i=0}^n f_i'(z)\overline{f_i'(w)}. \end{cases}$$

In [34, Theorem 1.1] a particular subfamily of random functions (1.10) was studied which is random polynomials of the form

$$P_n(z) = \eta_0\varphi_0(z) + \eta_1\varphi_1(z) + \cdots + \eta_n\varphi_n(z), \tag{1.12}$$

[1] In fact, Edelman and Kostlan derive an expression for the real intensity function for any random vector $(\eta_0, \ldots, \eta_n)$ in terms of its joint probability density function and of $v(x)$.

where η_i are i.i.d. standard real Gaussian random variables and $\varphi_i(z)$ are orthonormal polynomials on the unit circle with real coefficients. In this case it can be easily shown using Christoffel-Darboux formula (1.4), see [34, Theorem 1.1], that (1.11) can be rewritten as

$$\rho_n(x) = \frac{1}{\pi}\frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|}, \quad h_{n+1}(x) := \frac{(1-x^2)b_{n+1}'(x)}{1-b_{n+1}^2(x)}, \quad b_{n+1}(x) := \frac{\varphi_{n+1}(x)}{\varphi_{n+1}^*(x)}, \tag{1.13}$$

where, as before $\varphi_{n+1}^*(x) := x^{n+1}\varphi_{n+1}(1/x)$ is the reciprocal polynomial (there is no need for conjugation as all the coefficients are real). When μ is the normalized arclength measure on the unit circle, it is elementary to see that $\varphi_m(z) = z^m$ and therefore (1.13) recovers (1.6). Then the paper proceeds with several interesting results some of which we provide here. It was shown that when $m^p|\alpha_m|$ is a bounded sequence for some $p > 3/2$, estimate (1.7) remains valid. It was also shown that when $\alpha_k \to 0$ as $k \to \infty$, then

$$\lim_{n\to\infty} \rho_n^{(1,0)}(x) = \frac{1}{\pi}\frac{1}{|1-x^2|}$$

locally uniformly on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, and that

$$\lim_{n\to\infty} \rho_n^{(0,1)}(z) = \frac{1}{\pi}\frac{1}{(1-|z|^2)^2}\sqrt{1-\left|\frac{1-|z|^2}{1-z^2}\right|^2}$$

locally uniformly in $\overline{\mathbb{C}}\setminus(\mathbb{R}\cup\mathbb{T})$ as $n\to\infty$, where $\rho_n^{(1,0)}(z)$ and $\rho_n^{(0,1)}(z)$ are functions such that

$$\mathbb{E}\Big[N_n(\Omega)\Big] = \int_{\Omega\cap\mathbb{R}} \rho_n^{(1,0)}(x)\mathrm{d}x + \int_\Omega \rho_n^{(0,1)}(z)\mathrm{d}A,$$

If we take η_n's in (1.10) to be i.i.d. *complex* Gaussian random variables, then it was shown by Edelman and Kostlan [32], see also [35], that

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n(z)\mathrm{d}A, \tag{1.14}$$

where $\mathrm{d}A$ is the area measure and the intensity function $\rho_n(z)$ is given by

$$\rho_n(z) := \frac{1}{\pi}\frac{K_{n+1}^{(1,1)}(z,z)K_{n+1}(z,z) - |K_{n+1}^{(1,0)}(z,z)|^2}{\big(K_{n+1}(z,z)\big)^2}. \tag{1.15}$$

In this case, it was shown in [36, Corollary 1.2] that when $f_i = \varphi_i$, where $\{\varphi_i\}$ are OPUC associated to conjugate-symmetric measure μ and $|z| \neq 1$, the intensity function (1.15) reduces to

$$\rho_n(z) = \frac{1}{\pi}\frac{1-|k_n(z)|^2}{(1-|z|^2)^2}, \quad k_n(z) := \frac{(1-|z|^2)b_n'(z)}{1-|b_n(z)|^2}, \quad b_n(z) := \frac{\varphi_{n+1}(z)}{\varphi_{n+1}^*(z)}.$$

Moreover, when $\alpha_k \to 0$ as $k \to \infty$ and $\{\alpha_k\} \subset (-1,1)$, it holds that

$$\lim_{n\to\infty} \rho_n(z) = \frac{1}{\pi}\frac{1}{(1-|z|^2)^2}$$

locally uniformly in $\mathbb{C} \setminus \mathbb{T}$.

Inspired by all the above result, we write this dissertation with focus on the behavior of zeros of random polynomials spanned by OPUC. We study the behavior of the intensity function of the derivatives of random polynomials spanned by OPUC and obtain asymptotic expansion of the expected number of zeros in special cases of real random polynomials spanned by OPUC.

The layout of the document is as follows. In Chapter 2 we study zero distribution of random polynomials spanned by the derivatives of orthonormal polynomials in the unit circle with i.i.d Gaussian random variables. In Chapter 3 we study asymptotic behavior of the expected number of real zeros of random polynomials that are spanned by orthonormal polynomials in the unit circle with independent identically distributed standard Gaussian random variables. Finally, in Chapter 4 we study asymptotic behavior of the expected number of real zeros of random polynomials that are spanned by orthonormal Geronimus polynomials with i.i.d. standard Gaussian random variables.

2. ON THE INTENSITY FUNCTIONS OF THE DERIVATIVES OF RANDOM POLYNOMIALS SPANNED BY OPUC

Let $\{\varphi_i\}_{i=0}^{\infty}$ be a sequence of orthonormal polynomials on the unit circle with respect to a probability measure μ. We study zero distribution of random polynomials of the from

$$P_{n,m}(z) := \eta_0 \varphi_m^{(m)}(z) + \eta_1 \varphi_{m+1}^{(m)}(z) + \cdots + \eta_n \varphi_{n+m}^{(m)}(z)$$

where m is a non-negative integer, $\{\eta_n\}_{n=0}^{\infty}$ is a sequence of i.i.d. Gaussian random variables. We deduce the limiting value of the density functions in $\mathbb{D}$. We also provide results that estimate the expected number of zeros of $P_{n,m}$ in shrinking neighborhoods of compact subsets of $\mathbb{T}$.

2.1 Introduction

In this chapter we study zero distribution of random polynomials of the from

$$P_{n,m}(z) := \eta_0 \varphi_m^{(m)}(z) + \eta_1 \varphi_{m+1}^{(m)}(z) + \cdots + \eta_n \varphi_{n+m}^{(m)}(z) \tag{2.1}$$

as $n \to \infty$, where m is a non-negative integer, $\{\eta_n\}_{n=0}^{\infty}$ is a sequence of i.i.d. Gaussian random variables, and $\{\varphi_n(z)\}_{n=0}^{\infty}$ is a sequence of orthonormal polynomials on the unit circle $\mathbb{T}$.

2.2 Main Results

In this chapter we are interested in $N_n(\Omega)$, the random variable counting the number of zeros of $P_n(z)$ in a Jordan region $\Omega \subset \mathbb{C}$. More precisely, we will be interested in the limiting behavior of the intensity functions in Ω as well as the expectation $\mathbb{E}[N_n(\Omega)]$. We consider two cases:

(i) η_n's are i.i.d. complex Gaussian random variables;

(ii) η_n's are i.i.d. real Gaussian random variables and α_n's in (1.2) are real.

Clearly, in the second case polynomials $P_{n,m}(z)$ are real-valued on the real line and therefore one can expect some number of zeros to accumulate there.

We shall consider measures of orthogonality μ from the following three classes:

- UST-regular class (Ullman-Stahl-Totik) is defined by the condition $\lim_{n\to\infty} \kappa_n^{1/n} = 1$, see (1.3);
- Nevai's class is defined by the condition $\lim_{n\to\infty} \alpha_n = 0$;
- Szegő's class is defined by the condition $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$.

Notice that each subsequent class is a subset of the previous one. Nevai's class can be characterized by the condition

$$\lim_{n\to\infty} b_n(z) = 0, \quad b_n(z) := \varphi_n(z)/\varphi_n^*(z), \tag{2.2}$$

locally uniformly in $\mathbb{D}$, see [3, Theorem 1.7.4]. The Szegő class is characterized by the fact that

$$\int \log \mu'(z)|\mathrm{d}z| > -\infty, \quad \mathrm{d}\mu(z) = \mu'(z)|\mathrm{d}z| + \mathrm{d}\mu_s(z),$$

where $\mu'(z)$ is the Radon-Nikodym derivative of μ with respect to arclength measure $|\mathrm{d}z|$ and μ_s is singular with respect to $|\mathrm{d}z|$, see [3, Eq. (1.1.19)]. In this case it is known [3, Theorem 2.4.1] that

$$\lim_{n\to\infty} \varphi_n^*(z) = D_\mu^{-1}(z), \quad D_\mu(z) := \exp\left\{\int_{\mathbb{T}} \frac{\xi+z}{\xi-z}\log\mu'(\xi)\frac{|\mathrm{d}\xi|}{4\pi}\right\}, \tag{2.3}$$

locally uniformly in $\mathbb{D}$, where $D_\mu(z)$ is now known as the Szegő function of μ.

2.2.1 Case (i): Complex Roots

In this subsection we assume that η_n are i.i.d. *complex* Gaussian random variables. It was shown by Edelman and Kostlan [32], see also [35], that

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_{n,m}(z)\mathrm{d}A, \tag{2.4}$$

where $\mathrm{d}A$ is the area measure and the intensity function $\rho_{n,m}(z)$ is given by

$$\rho_{n,m}(z) := \frac{1}{\pi} \frac{K_{n+m+1}^{(m+1,m+1)}(z,z) K_{n+m+1}^{(m,m)}(z,z) - |K_{n+m+1}^{(m+1,m)}(z,z)|^2}{\left(K_{n+m+1}^{(m,m)}(z,z)\right)^2}, \tag{2.5}$$

where $K_{n+1}^{(j,k)}(z,w) := \partial_z^j \overline{\partial}_w^k K_{n+1}(z,w)$.

To describe the asymptotic behavior of the zeros of $P_{n,m}(z)$, define

$$\Omega(S,\tau_1,\tau_2) := \left\{ rz : z \in S, r \in \left(1 + \frac{\tau_1}{2n}, 1 + \frac{\tau_2}{2n}\right) \right\}$$

for any real numbers $-\infty < \tau_1 < \tau_2 < \infty$ and any set $S \subseteq \mathbb{T}$. Further, set $H(\tau) := \dfrac{e^\tau - 1}{\tau}$ and observe that

$$H^{(k)}(\tau) = \frac{e^\tau p_k(\tau) - (-1)^k k!}{\tau^{k+1}},$$

where $p_0(\tau) \equiv 1$ and $p_k(\tau) := (\tau - k)p_{k-1}(\tau) + \tau p'_{k-1}(\tau)$ is a monic polynomial of degree k. In particular, it holds that

$$\lim_{\tau\to\infty} \frac{H^{(k+1)}(\tau)}{H^{(k)}(\tau)} = 1 \quad \text{and} \quad \lim_{\tau\to-\infty} \frac{H^{(k+1)}(\tau)}{H^{(k)}(\tau)} = 0$$

for any $k \geq 0$. Then the following theorem takes place.

Theorem 2.2.1. *Let $P_{n,m}(z)$ be given by (2.1) with UST-regular μ and η_n's being i.i.d. complex Gaussian random variables. Let S be a compact subset of $\mathbb{T}$. Assume, in addition, that μ is absolutely continuous with respect to the arclength measure on an open set containing S and its Radon-Nikodym derivative is positive and continuous at each point of S. Then it holds that*

$$\frac{1}{n}\mathbb{E}\big[N_n(\Omega(S,\tau_1,\tau_2))\big] \to \frac{|S|}{2\pi}\left(\frac{H^{(2m+1)}(\tau_2)}{H^{(2m)}(\tau_2)} - \frac{H^{(2m+1)}(\tau_1)}{H^{(2m)}(\tau_1)}\right) \quad \textit{as} \quad n \to \infty. \tag{2.6}$$

As far as the scaling limits away from the unit circle are concerned, it was shown in [36, Corollary 1.2] that if μ is in Nevai's class, then

$$\lim_{n\to\infty} \rho_{n,0}(z) = \frac{1}{\pi} \frac{1}{(1-|z|^2)^2}$$

locally uniformly in $\mathbb{C}\setminus\mathbb{T}$. Hence, the expected number of zeros of $P_{n,0}(z)$ in this case approaches a finite non-zero limit for any $\Omega\subset\mathbb{C}\setminus\mathbb{T}$ of non-zero area. To discuss what happens when $m>0$, we further assume that polynomials $\varphi_n(z)$ belong to Szegő's class. Given a Szegő measure μ, we can define

$$F_\mu(z) := -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{\log\mu'(\xi)}{(\xi-z)^2}\mathrm{d}\xi, \quad z\notin\mathbb{T}, \tag{2.7}$$

which is simply the derivative of the Cauchy integral of $-\log\mu'(\xi)$ or the logarithmic derivative of the reciprocal of the Szegő function of μ, see (2.3). Then the following theorem holds.

Theorem 2.2.2. *Let $P_{n,m}(z)$ be given by (2.1) with μ in Szegő's class and η_n's being i.i.d. complex Gaussian random variables. Then*

$$\lim_{n\to\infty}\rho_{n,m}(z) = R_{\mu,m}(z), \quad R_{\mu,m}(z) := \frac{1}{\pi}\partial_{\overline{z}}\partial_z\log S_{\mu,m}(z) \tag{2.8}$$

locally uniformly in $\mathbb{D}$, where $S_{\mu,0}(z)=S(z):=(1-|z|^2)^{-1}$ and

$$S_{\mu,m+1}(z) := \partial_{\overline{z}}\partial_z S_{\mu,m}(z) + \overline{F_\mu(z)}\partial_z S_{\mu,m}(z) + F_\mu(z)\partial_{\overline{z}}S_{\mu,m}(z) + |F_\mu(z)|^2 S_{\mu,m}(z). \tag{2.9}$$

In particular,

$$R_{\mu,1}(z) = \frac{4}{\pi}\frac{S^2(z)}{1+|A_1(z)|^2}\left(1+\frac{\left|A_1^2(z)-A_2(z)\right|^2}{1+|A_1(z)|^2}\right), \tag{2.10}$$

where $A_1(z) := \overline{z}+S^{-1}(z)F_\mu(z)$ and $A_2(z) := S^{-2}(z)\Big(F_\mu^2(z)+F_\mu'(z)\Big)/2$.

Notice that when μ is a constant multiple of a Lebesgue measure on $\mathbb{T}$, it holds that $F_\mu(z)\equiv 0$, $A_1(z)=\overline{z}$, $A_2(z)\equiv 0$, and therefore

$$R_{|d\xi|,1}(z) = \frac{4}{\pi}\frac{1}{1-|z|^2}\frac{1-|z|^6}{(1-|z|^4)^2}.$$

2.2.2 Case (ii): Complex Roots

In this subsection we assume that η_n are i.i.d. *real* Gaussian random variables and $\{\alpha_n\}_{n=0}^{\infty} \subset (-1,1)$ (that is, measure μ is conjugate-symmetric, and therefore the polynomials $\varphi_n(z)$ have real coefficients). In this case it is known [21, 33, 37] that

$$\mathbb{E}\Big[N_n(\Omega)\Big] = \int_{\Omega\cap\mathbb{R}} \rho_{n,m}^{(1,0)}(x)\mathrm{d}x + \int_{\Omega} \rho_{n,m}^{(0,1)}(z)\mathrm{d}A, \tag{2.11}$$

where the intensity function of the real roots is given by

$$\rho_{n,m}^{(1,0)}(x) = \frac{1}{\pi}\frac{\sqrt{K_{n+m+1}^{(m+1,m+1)}(x,x)K_{n+m+1}^{(m,m)}(x,x) - K_{n+m+1}^{(m+1,m)}(x,x)^2}}{K_{n+m+1}^{(m,m)}(x,x)} \tag{2.12}$$

and the intensity function of complex roots is given by

$$\begin{aligned}\rho_{n,m}^{(0,1)}(z) = \frac{1}{\pi}&\frac{K_{n+m+1}^{(m+1,m+1)}(z,z)}{\Big(K_{n+m+1}^{(m,m)}(z,z)^2 - |K_{n+m+1}^{(m,m)}(z,\overline{z})|^2\Big)^{1/2}} - \\ &\frac{1}{\pi}\frac{K_{n+m+1}^{(m,m)}(z,z)\Big(|K_{n+m+1}^{(m+1,m)}(z,z)|^2 + |K_{n+m+1}^{(m+1,m)}(z,\overline{z})|^2\Big)}{\Big(K_{n+m+1}^{(m,m)}(z,z)^2 - |K_{n+m+1}^{(m,m)}(z,\overline{z})|^2\Big)^{3/2}} + \\ &\qquad\frac{2}{\pi}\frac{\mathrm{Re}\Big(K_{n+m+1}^{(m,m)}(z,\overline{z})K_{n+m+1}^{(m+1,m)}(z,z)K_{n+m+1}^{(m+1,m)}(\overline{z},z)\Big)}{\Big(K_{n+m+1}^{(m,m)}(z,z)^2 - |K_{n+m+1}^{(m,m)}(z,\overline{z})|^2\Big)^{3/2}}.\end{aligned} \tag{2.13}$$

Then the following theorem takes place.

Theorem 2.2.3. *Let $P_{n,m}(z)$ be given by (2.1) with a conjugate-symmetric measure μ in Nevai's class and η_n's being i.i.d. real Gaussian random variables. Let $S \subset \mathbb{T} \setminus \{\pm 1\}$ be compact. Assume μ satisfies the conditions of Theorem 2.2.1 around S. Then (2.6) remains valid.*

2.2.3 Real Roots

Below, we keep the assumptions of the previous subsection and consider the behavior of the polynomials $P_{n,m}(z)$ on the real line. Given a measurable set $\Omega \subset \mathbb{R}$, it follows from (2.11) that

$$\mathbb{E}\Big[N_n(\Omega)\Big] = \int_{\Omega\cap\mathbb{R}} \rho_{n,m}^{(1,0)}(x)\mathrm{d}x,$$

where $\rho_{n,m}^{(1,0)}(x)$ is given by (2.12). We start by considering what happens around ± 1.

Theorem 2.2.4. *Let $P_{n,m}(z)$ be given by (2.1) with conjugate-symmetric UST-regular measure μ and η_n's being i.i.d. real Gaussian random variables. Assume, in addition, that μ is absolutely continuous with respect to the arclength measure on an arc containing 1 and its Radon-Nikodym derivative is positive and continuous at 1. Then it holds that*

$$\mathbb{E}\left[N_n\left(\left(1+\frac{\tau_1}{2n}, 1+\frac{\tau_2}{2n}\right)\right)\right] \to \frac{1}{2\pi}\int_{\tau_1}^{\tau_2}\sqrt{\left(\frac{H^{(2m+1)}(\tau)}{H^{(2m)}(\tau)}\right)'}\mathrm{d}\tau \quad \text{as} \quad n\to\infty.$$

An analogous result can be stated around -1.

It was shown in [34, Corollary 1.3] that if μ is in Nevai's class, then

$$\lim_{n\to\infty}\rho_{n,0}^{(1,0)}(x) = \frac{1}{\pi}\frac{1}{|1-x^2|}$$

locally uniformly on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. When $m>0$, the following result takes place.

Theorem 2.2.5. *Let $P_{n,m}(z)$ be given by (2.1) with a conjugate-symmetric measure μ in Szegő's class and η_n's being i.i.d. real Gaussian random variables. Then*

$$\lim_{n\to\infty}\rho_{n,m}^{(1,0)}(x) = \sqrt{R_{\mu,m}(x)/\pi}$$

locally uniformly on $(-1,1)$, where $R_{\mu,m}(x)$ are the same functions as in Theorem 2.2.2.

Observe that for a conjugate-symmetric measure μ, the function $F_\mu(z)$ satisfies $F_\mu(\overline{z}) = \overline{F_\mu(z)}$.

2.3 Proof of the Main Results

Proof of Theorems 2.2.1, 2.2.3, and 2.2.4

Proof of Theorem 2.2.1

Observe that

$$\begin{aligned}\frac{1}{n}\mathbb{E}\Big[N_n(\Omega(S,\tau_1,\tau_2))\Big] &= \frac{1}{n}\iint_{\Omega(S,\tau_1,\tau_2)}\rho_{n,m}(z)\mathrm{d}A \\ &= \frac{1}{2n^2}\int_S\int_{\tau_1}^{\tau_2}\rho_{n,m}\left(z\left(1+\frac{\tau}{2n}\right)\right)\left(1+\frac{\tau}{2n}\right)\mathrm{d}\tau|\mathrm{d}z| \end{aligned} \tag{2.14}$$

by (2.4). Hence, it is enough to show that

$$\frac{1}{2n^2}\rho_{n,m}\left(z\left(1+\frac{\tau}{2n}\right)\right)\to\frac{1}{2\pi}\left(\frac{H^{(2m+1)}(\tau)}{H^{(2m)}(\tau)}\right)' \tag{2.15}$$

uniformly for $z\in S$ and τ on compact subsets of the real line. It follows from [38, Theorem 6.3] and Cauchy's integral formula that

$$\lim_{n\to\infty}\frac{z^{k-j}}{n^{k+j}}\frac{K_n^{(k,j)}(z_{n,u},z_{n,\bar{v}})}{K_n(z,z)} = H^{(k+j)}(u+v) \tag{2.16}$$

uniformly for $z\in S$ and u,v on compact subsets of $\mathbb{C}$, where $z_{n,a}:=z\left(1+a/n\right)$. By using the above limits with $u=v=\tau/2$ and plugging them in (2.5), we get (2.15) (recall that τ real and so is $H(\tau)$ in this case).

Proof of Theorem 2.2.4

It follows from (2.11) that formula (2.14) needs to be replaced by

$$\mathbb{E}\left[N_n\left(\left(1+\frac{\tau_1}{2n},1+\frac{\tau_2}{2n}\right)\right)\right] = \int_{1+\tau_1/(2n)}^{1+\tau_2/(2n)}\rho_{n,m}^{(1,0)}(x)\mathrm{d}x = \frac{1}{2n}\int_{\tau_1}^{\tau_2}\rho_{n,m}^{(1,0)}\left(1+\frac{\tau}{2n}\right)\mathrm{d}\tau.$$

It also can be clearly seen from (2.5) and (2.12) that $\rho_{n,m}^{(1,0)}(x)=\sqrt{\rho_{n,m}(x)/\pi}$. The claim of the theorem now follows from (2.15).

Proof of Theorem 2.2.3

Observe that in this case (2.14) remains valid with $\rho_{n,m}(z)$ replaced by $\rho^{(0,1)}_{n,m}(z)$. Thus, we again need to prove that the integrand in (2.14) has limit (2.15). As mentioned before, measures in Nevai's class are UST-regular and therefore (2.16) remains valid.

Recall that μ is absolutely continuous with respect to $|\mathrm{d}z|$ in some neighborhood of S. Let $\mu'(z)$ be its Radon-Nikodym derivative. According to the conditions of the theorem, $\mu'(z)$ is positive and continuous on S. Hence, there exists an open cover of S by finitely many subarcs, which we denote by I, such that $c^{-1}\leq|\mu'(z)|\leq c$ for $z\in I$ and some $c>1$. Let μ_- and μ_+ be measures that coincide with μ on $\mathbb{T}\setminus I$ and that are defined by $c^{-1}|\mathrm{d}z|$ and $c|\mathrm{d}z|$, respectively, on I. These measures are UST-regular by [39, Theorem 5.3.3]. Let J be a cover of S constructed similarly to I and such that $\overline{J}\subset I$ and $\pm1\notin\overline{J}$. Then it follows from [38, Theorem 3.1] that

$$C^{-1}n\leq|K_n^{\mu_+}(z,z)|\leq|K_n^{\mu}(z,z)|\leq|K_n^{\mu_-}(z,z)|\leq Cn,\quad z\in\overline{J},\tag{2.17}$$

for all n large enough and some constant $C>1$ independent of z and n (the inequality for the kernels is well known and follows from the identity $K_n^{-1}(z,z)=\inf_{p_{n-1}(z)=1}\int|p_{n-1}|^2\mathrm{d}\mu$, where the infimum is taken over all polynomials of degree at most $n-1$). Thus, given (2.13), (2.16), and (2.17), it is enough to show that

$$\lim_{n\to\infty}n^{-(k+j+1)}|K_n^{(k,j)}(z_{n,a},\overline{z}_{n,a})|=0\tag{2.18}$$

uniformly for $z\in S$ and a on compact subsets of $\mathbb{C}$.

Using holomorphy of $K_n^{(k,j)}(z,\overline{w})$ in z and w, we get from Cauchy's integral formula that

$$\left|K_n^{(k,j)}(u,\overline{v})\right|=\left|\int_{C_n(u)}\int_{C_n(v)}\frac{K_n(\xi,\overline{\zeta})}{(\xi-u)^{k+1}(\zeta-v)^{j+1}}\frac{\mathrm{d}\xi\mathrm{d}\zeta}{(2\pi\mathrm{i})^2}\right|\leq \frac{|C_n(u)||C_n(v)|}{(2\pi)^2}\frac{\max_{\xi\in C_n(u),\zeta\in C_n(v)}|K_n(\xi,\overline{\zeta})|}{\mathrm{dist}(u,C_n(u))^{k+1}\mathrm{dist}(v,C_n(v))^{j+1}},$$

where $C_n(u), C_n(v)$ are Jordan curves encircling u, v and $|C_n(u)|, |C_n(v)|$ are their arclength. Let $u = z_{n,a}$, $v = \overline{z}_{n,a}$ and $C_n(u) = C_n(v) = C_n = \{|z - \xi| = 2R/n\}$, where $|a| < R$. Then it holds that

$$n^{-(k+j+1)}\left|K_n^{(k,j)}(z_{n,a}, \overline{z}_{n,a})\right| \leq \frac{\max_{\zeta,\xi\in C_n}|K_n(\xi,\overline{\zeta})|}{nR^{k+j}}. \tag{2.19}$$

Recall that in the considered case the polynomials $\varphi_n(z)$ have real coefficients. Therefore, it follows from Christoffel-Darboux formula (1.4) that

$$\max_{\zeta,\xi\in C_n}|K_n(\xi,\overline{\zeta})| = \max_{\zeta,\xi\in C_n}\left|\frac{\varphi_n^*(\xi)\varphi_n^*(\zeta) - \varphi_n(\xi)\varphi_n(\zeta)}{1-\xi\zeta}\right| \leq \frac{\max_{\xi\in C_n}\left(|\varphi_n^*(\xi)|^2 + |\varphi_n(\xi)|^2\right)}{\operatorname{dist}(\pm 1, \overline{J})}$$

for all n large enough. By observing that $|\varphi_n^*(z)| = |\varphi_n(z)|$ for $z \in \mathbb{T}$ and applying Bernstein-Walsh inequality on each subarc of J, see [38, Lemma 6.1], we get that

$$\max_{\xi\in C_n}\left(|\varphi_n^*(\xi)|^2 + |\varphi_n(\xi)|^2\right) \leq \tilde{C}\max_{z\in\overline{J}}|\varphi_n(z)|^2$$

for some constant $\tilde{C}$ independent of n (it does depend on J and R in the definition of C_n). Plugging the last two estimates into (2.19) and using (2.17) gives us

$$n^{-(k+j+1)}\left|K_n^{(k,j)}(z_{n,a}, \overline{z}_{n,a})\right| \leq \hat{C}\max_{z\in\overline{J}}\frac{1}{n}|\varphi_n(z)|^2 \leq \hat{C}C\max_{z\in\overline{J}}\frac{|\varphi_n(z)|^2}{K_n(z,z)} \leq \hat{C}C\max_{z\in\mathbb{T}}\frac{|\varphi_n(z)|^2}{K_n(z,z)}$$

for some constant $\hat{C}$. Since the measure μ is in Nevai's class, it follows from [40, Theorem 4] that $\max_{z\in\mathbb{T}}|\varphi_n(z)|^2 K_n^{-1}(z,z) \to 0$ as $n \to \infty$, which finishes the proof of (2.18).

Proof of Theorems 2.2.2 and 2.2.5

Proof of Theorem 2.2.2

Recall that $K_n(z,\overline{w}) = \sum_{i=0}^{n-1}\varphi_i(z)\overline{\varphi_i(\overline{w})}$ is an analytic function in both variables. Assume that

$$\frac{K_n^{(m,m)}(z,\overline{w})}{\varphi_n^*(z)\overline{\varphi_n^*(\overline{w})}} \to S_m(z,w) \tag{2.20}$$

locally uniformly for $(z,w)\in\mathbb{D}^2$. Notice that

$$S_m(z,\overline{w})=\overline{S_m(w,\overline{z})} \quad \text{and therefore} \quad S_m(z,\overline{z})\geq 0. \tag{2.21}$$

Since uniform convergence of analytic functions implies uniform convergence of their derivatives, it follows from (2.3) and the definition of $F_\mu(z)$ in (2.7) that

$$\frac{K_n^{(m+1,m)}(z,\overline{w})}{\varphi_n^*(z)\overline{\varphi_n^*(\overline{w})}}\to S_m^{(1,0)}(z,w)+S_m(z,w)F_\mu(z)$$

locally uniformly for $(z,w)\in\mathbb{D}^2$. Furthermore, we similarly can deduce that (2.20) holds with m replaced by $m+1$, where

$$S_{m+1}(z,w):=S_m^{(1,1)}(z,w)+S_m^{(1,0)}(z,w)\overline{F_\mu(\overline{w})}+S_m^{(0,1)}(z,w)F_\mu(z)+S_m(z,w)F_\mu(z)\overline{F_\mu(\overline{w})}. \tag{2.22}$$

In fact, since Szegő's measures are in Nevai's class, it follows from (2.2) and (**??**) that

$$\frac{K_n(z,\overline{w})}{\varphi_n^*(z)\overline{\varphi_n^*(\overline{w})}}=\frac{1-b_{n+1}(z)\overline{b_{n+1}(\overline{w})}}{1-zw}\to\frac{1}{1-zw}=:S_0(z,w) \tag{2.23}$$

locally uniformly for $(z,w)\in\mathbb{D}^2$. Therefore, (2.20) does indeed hold for all $m\geq 0$ with $S_m(z,w)$ defined inductively via (2.22).

Set $S_{\mu,m}(z):=S_m(z,\overline{z})$. Then $\partial_{\overline{z}}S_{\mu,m}=S_m^{(0,1)}$, $\partial_z S_{\mu,m}=S_m^{(1,0)}$, and $\partial_{\overline{z}}\partial_z S_{\mu,m}=S_m^{(1,1)}$. Hence, one can readily see from (2.22) and (2.23) that $S_{\mu,m}(z)$ satisfy (2.9). It also follows from (2.21) that

$$\partial_{\overline{z}}S_{\mu,m}(z)=\partial_{\overline{z}}S_m(z,\overline{z})=\partial_{\overline{z}}\overline{S_m(z,\overline{z})}=\overline{\partial_z S_m(z,\overline{z})}=\overline{\partial_z S_{\mu,m}(z)}. \tag{2.24}$$

Therefore, we get from (2.5) that

$$\rho_{n,m}(z)\to\frac{1}{\pi}\frac{(\partial_{\overline{z}}\partial_z S_{\mu,m}(z))S_{\mu,m}(z)-(\partial_z S_{\mu,m}(z))(\partial_{\overline{z}}S_{\mu,m}(z))}{S_{\mu,m}^2(z)}=\frac{1}{\pi}\partial_{\overline{z}}\partial_z\log S_{\mu,m}(z) \tag{2.25}$$

locally uniformly in $\mathbb{D}$, which proves (2.8).

It only remains to prove (2.10). We have that

$$S_{\mu,1}(z) = S^3(z)\Big(1+|A(z)|^2\Big), \quad A(z) := A_1(z) = \overline{z} + (1-|z|^2)F_\mu(z).$$

Write $\bar{A}(z) := \overline{A(z)}$. Then, we have that

$$\begin{aligned} S^{-3}\partial_z S_{\mu,1} &= 3\overline{z}S\Big(1+|A|^2\Big) + \Big((\partial_z A)\bar{A} + A(\partial_z \bar{A})\Big) = 3\overline{z}S + A\Big(\partial_z \bar{A} + \overline{z}S\bar{A}\Big) + \bar{A}B \\ &= S\Big(3\overline{z} + A\Big) + \bar{A}B, \quad B := \partial_z A + 2\overline{z}SA. \end{aligned} \tag{2.26}$$

We further have that

$$S^{-3}\partial_{\overline{z}}\partial_z S_{\mu,1} = S^2(12|z|^2 + 4zA) + S(3+\partial_{\overline{z}}A) + 3zS\bar{A}B + \overline{(\partial_z A)}B + \bar{A}(\partial_{\overline{z}}B).$$

Notice that

$$\partial_{\overline{z}}A = S - zSA \quad \text{and} \quad \overline{(\partial_z A)}B = |B|^2 - 2zS\bar{A}B.$$

Therefore,

$$\begin{aligned} S^{-3}\partial_{\overline{z}}\partial_z S_{\mu,1} &= S^2(12|z|^2 + 3zA) + S(3+S) + |B|^2 + zS\bar{A}B + \bar{A}(\partial_{\overline{z}}B) \\ &= S^2(4+9|z|^2+3zA) + |B|^2 + zS\bar{A}B + \bar{A}(\partial_{\overline{z}}B) \\ &= 4S^2 + S^2|3\overline{z}+A|^2 + |B|^2 - S^2(3\overline{z}\bar{A} + |A|^2) + zS\bar{A}B + \bar{A}(\partial_{\overline{z}}B). \end{aligned}$$

Moreover,

$$\partial_{\overline{z}}B = \partial_{\overline{z}}\partial_z A + 2SA + 2|z|^2S^2A + 2\overline{z}S\partial_{\overline{z}}A = \partial_{\overline{z}}\partial_z A + 2SA + 2\overline{z}S^2.$$

Next, one can verify that

$$\partial_{\overline{z}}\partial_z A = -zS\partial_z A + S^2(\overline{z}-A) = -zSB + 2|z|^2S^2A + S^2(\overline{z}-A) = -zSB - 2SA + S^2(\overline{z}+A)$$

That is,

$$\partial_{\overline{z}}B = -zSB + S^2(3\overline{z}+A),$$

which, in turn, leads to

$$S^{-3}\partial_{\overline{z}}\partial_z S_{\mu,1} = 4S^2 + S^2|3\overline{z} + A|^2 + |B|^2 \tag{2.27}$$

Finally, by combining (2.25) with (2.26) and (2.27), and recalling (2.24) we get that $\pi\rho_{n,1}(z)$ converges locally uniformly in $\mathbb{D}$ to

$$\frac{(4S^2 + S^2|3\overline{z} + A|^2 + |B|^2)(1 + |A|^2) - \left|S\Big(3\overline{z} + A\Big) + \bar{A}B\right|^2}{(1 + |A|^2)^2} = \frac{4S^2}{1 + |A|^2} + \frac{\left|SA(3\overline{z} + A) - B\right|^2}{(1 + |A|^2)^2}.$$

Finally, observe that

$$\begin{aligned} A(3\overline{z} + A) - BS^{-1} &= A(\overline{z} + A) - S^{-1}\partial_z A = A^2 + 2\overline{z}A - \overline{z}^2 - S^{-2}F'_\mu \\ &= 2A^2 - (A - \overline{z})^2 - S^{-2}F'_\mu = 2A^2 - S^{-2}(F_\mu^2 + F'_\mu) \end{aligned}$$

which yields (2.10).

Proof of Theorem 2.2.5

Since $\rho_{n,m}^{(1,0)}(x) = \sqrt{\rho_{n,m}(x)/\pi}$ by (2.5) and (2.12), the claim follows immediately from Theorem 2.2.2.

3. AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF REAL RANDOM POLYNOMIALS SPANNED BY OPUC

A version of this chapter is published in [41].

Let $\{\varphi_i\}_{i=0}^{\infty}$ be a sequence of orthonormal polynomials on the unit circle with respect to a positive Borel measure μ that is symmetric with respect to conjugation. We study asymptotic behavior of the expected number of real zeros, say $\mathbb{E}_n(\mu)$, of random polynomials (1.12). In this chapter we generalize (1.8) to the case where μ is absolutely continuous with respect to arclength measure and its Radon-Nikodym derivative extends to a holomorphic non-vanishing function in some neighborhood of the unit circle. In this case we show that $\mathbb{E}_n(\mu)$ admits an expansion similar to (1.8), where the coefficients A_p do depend on the measure μ for $p \geq 1$ while A_0 is still given by (1.9).

3.1 Introduction and Main Results

In this chapter we investigate the asymptotic behavior of the expected number of real zeros of random polynomials (1.12) for a special class of measures μ. Our approach follows the one of Wilkins [6] and is based on asymptotic analysis of quantities in (1.13).

Theorem 3.1.1 (Main result)**.** *Let $P_n(z)$ be given by (1.1)–(1.12), for a conjugate-symmetric measure μ that is absolutely continuous with respect to the arclength measure and is such that $\mu'(\xi)$, the respective Radon-Nikodym derivative, extends to a holomorphic non-vanishing function in some neighborhood of the unit circle. Then $\mathbb{E}_n(\mu)$, the expected number of real zeros of $P_n(z)$, satisfies*

$$\mathbb{E}_n(\mu) = \frac{2}{\pi}\log(n+1) + A_0 + \sum_{p=1}^{N-1} A_p^{\mu}(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right)$$

for any integer N and all n large, where $\mathcal{O}_N(\cdot)$ depends on N, but is independent of n, A_0 is given by (1.9), and A_p^{μ}, $p \geq 1$, are some constants that do depend on μ.

Clearly, the above result generalizes (1.8), where $\mathrm{d}\mu(\xi) = |\mathrm{d}\xi|/(2\pi)$.

In the next section, we introduce some auxiliary estimates that are helpful in proving the mean results.

3.2 Auxiliary Estimates

In this section we gather some auxiliary estimates of quantities involving orthonormal polynomials $\varphi_m(z)$. First of all, recall (1.2), where, in our case, the recurrence coefficients $\{\alpha_m\}$ belong to the interval $(-1,1)$ due to conjugate symmetry of the measure μ. In what follows we denote by $\rho<1$ the smallest number such that $\mu'(\xi)$ is non-vanishing and holomorphic in the annulus $\{\rho<|z|<1/\rho\}$.

With a slight abuse of notation we shall denote various constant that depend on μ and possibly additional parameters r,s by the same symbol $C_{\mu,r,s}$ understanding that the actual value of $C_{\mu,r,s}$ might be different for different occurrences, but it never depends on z or n.

Lemma 3.2.1. *It holds that*

$$|h_{n+1}(x)|\leq C_\mu(n+1)e^{-\sqrt{n+1}}, \quad |x|\leq 1-(n+1)^{-1/2}.$$

Proof. Using the recurrence relations for polynomials $\Phi_m(z)$, one can readily verify that

$$h_{n+1}(z)=(1-z^2)\frac{(zb_n(z))'}{1-(zb_n(z))^2}.$$

Therefore, the last estimate in [34, Section 3.3] implies that

$$|h_{n+1}(x)|\leq C_\mu|\big(xb_n(x)\big)'|, \quad |x|\leq 1-(n+1)^{-1/2}.$$

Moreover, it was shown in [34, Equation (48)] that

$$|\big(zb_n(z)\big)'|\leq C_\mu(n+1)\left(r^{n-m}+\sum_{i=m}^{\infty}|\alpha_i|\right), \quad |z|\leq r<1.$$

It is further known, see [42, Corollary 2], that the recurrence coefficients α_i satisfy

$$|\alpha_i|\leq C_{\mu,\rho-s}s^{i+1} \quad \Rightarrow \quad \sum_{i=m}^{\infty}|\alpha_i|\leq\frac{C_{\mu,s-\rho}s^m}{1-\rho}, \quad \rho<s<1,$$

where $C_{\mu,s-\rho}$ also depends on how close s is to ρ. Given a value of the parameter s, take m to be the integer part of $-\sqrt{n+1}/\log s$ and $r = 1 - 1/\sqrt{n+1}$. By combining the above three estimates, we deduce the desired inequality with a constant that depends on μ, $s-\rho$, and s. Optimizing the constant over s finishes the proof of the lemma. $\square$

Denote by $D(z)$ the Szegő function of μ, i.e.,

$$D(z) := \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}}\frac{\xi+z}{\xi-z}\log\mu'(\xi)|\mathrm{d}\xi|\right\}, \quad |z| \neq 1.$$

This function is piecewise analytic and non-vanishing. Denote by $D_{int}(z)$ the restriction of $D(z)$ to $|z|<1$ and by $D_{ext}(z)$ the restriction to $|z|>1$. It is known that both $D_{int}(z)$ and $D_{ext}(z)$ extend continuously to the unit circle and satisfy there

$$D_{int}(\xi)/D_{ext}(\xi) = \mu'(\xi), \quad |\xi| = 1.$$

Moreover, since $\mu'(\xi)$ extends to a holomorphic and non-vanishing function in the annulus $\rho < |z| < 1/\rho$, $D_{int}(z)$ and $D_{ext}(z)$ extend to holomorphic and non-vanishing functions in $|z| < 1/\rho$ and $|z| > \rho$, respectively. Hence, the scattering function

$$S(z) := D_{int}(z)D_{ext}(z), \quad \rho < |z| < 1/\rho,$$

is well defined and non-vanishing in this annulus. Since the measure μ is conjugate symmetric, it holds that $D(\bar{z}) = \overline{D(z)}$ and $D_{ext}(1/z) = 1/D_{int}(z)$. Thus, $|S(\xi)| = 1$ for $|\xi| = 1$ and $S(1) = 1$. For future use let us record the following straightforward facts.

Lemma 3.2.2. *There exist real numbers s_p, $p \geq 1$, such that*

$$\begin{aligned} S(z) &= 1 + \textstyle\sum_{p=1}^{M-1} s_p(1-z)^p + E_M(S;z) \\ S'(z) &= -\textstyle\sum_{p=0}^{M-1}(p+1)s_{p+1}(1-z)^p + E_M(S';z) \\ \log S(z) &= \textstyle\sum_{p=1}^{M-1} c_p(1-z)^p + E_M(\log S;z) \end{aligned}$$

for $|z-1|<T<1-\rho$ *and any integer* $M\geq 1$*, where the error terms satisfy*

$$\left|E_M(F;z)\right| \leq \frac{\|F\|_{|z-1|\leq T}}{1-|1-z|/T}\left(\frac{|1-z|}{T}\right)^M$$

and $c_p = s_p + \sum_{k=2}^{p}\frac{(-1)^{k-1}}{k}\sum_{j_1+\cdots+j_k=p} s_{j_1}\cdots s_{j_k}$*. Moreover,* $s_2 = s_1(s_1+1)/2$*. In particular,* $c_1 = s_1$ *and* $c_2 = s_1/2$*.*

Proof. Since $c_1 = s_1$ and $c_2 = s_2 - s_1^2/2$, we only need to show that $s_2 = s_1(s_1+1)/2$. It holds that $s_1 = -S'(1)$ and $s_2 = S''(1)/2$. Using the symmetry $1\equiv S(z)S(1/z)$, one can check that $S''(1) = S'(1)^2 - S'(1)$, from which the desired claim easily follows. □

Set $\tau := D_{ext}(\infty)$. It has been shown in [42, Theorem 1] that

$$\Phi_m(z) = \tau^{-1}z^m D_{ext}(z)\mathcal{E}_m(z) - \frac{\tau\mathcal{I}_m(z)}{D_{int}(z)}, \quad \rho<|z|<1/\rho, \tag{3.1}$$

for some recursively defined functions $\mathcal{E}_m(z),\mathcal{I}_m(z)$ holomorphic in the annulus $\rho<|z|<1/\rho$ that satisfy

$$\left|\mathcal{E}_m(z)-1\right| \leq \frac{C_{\mu,s}s^{2m}}{1/s-|z|} \quad \text{and} \quad \left|\mathcal{I}_m(z)\right| \leq \frac{C_{\mu,s}s^m}{|z|-s}, \quad \rho<s<|z|<1/s, \tag{3.2}$$

for some explicitly defined constant $C_{\mu,s}$, see [42, Equations (34)-(35)]. In particular, it follows from (3.1) that

$$b_{n+1}(z) = z^{n+1}S(z)H_n(z), \quad H_n(z) := \frac{\mathcal{E}_{n+1}(z) - \tau^2 z^{-(n+1)}S^{-1}(z)\mathcal{I}_{n+1}(z)}{\mathcal{E}_{n+1}(1/z) - \tau^2 z^{n+1}S(z)\mathcal{I}_{n+1}(1/z)}, \tag{3.3}$$

for $\rho<|z|<1/\rho$. It can be checked that the conjugate symmetry of μ yields real-valuedness of $H_n(z)$ on the real line. Bounds (3.2) also imply that $H_n(x)$ is close to 1 near $x=1$. More precisely, the following lemma holds.

Lemma 3.2.3. *It holds for any* $\rho<\rho_*<1$ *that*

$$|H_n(x)-1|, |\log H_n(x)| \leq (1-x)C_{\mu,\rho_*}e^{-\sqrt{n+1}}, \quad \rho_*\leq x\leq 1.$$

Moreover, it also holds that $|H_n'(x)|\leq C_{\mu,\rho_*}e^{-\sqrt{n+1}}$ *on the same interval.*

Proof. Define $W_n(z) := \mathcal{E}_{n+1}(z) - 1 - \tau^2 z^{-(n+1)} S^{-1}(z)\mathcal{I}_{n+1}(z)$ and choose $\rho < s < s_* < \rho_* < 1$. Since $S(z)$ is a fixed non-vanishing holomorphic function in the annulus $\rho < |z| < 1/\rho$, it follows from (3.2) that

$$|W_n(z)| \leq C_{\mu,s,s_*}\big(s/s_*\big)^n, \quad s_* \leq |z| \leq 1/s_*.$$

It further follows from the maximum modulus principle that

$$|W_n(z) - W_n(1/z)| \leq |1-z| C_{\mu,s,s_*}\big(s/s_*\big)^n, \quad s_* \leq |z| \leq 1/s_*,$$

where, as agreed before, the actual constants in the last two inequalities are not necessarily the same. Since $|\log(1+\zeta)| \leq 2|\zeta|$ for $|\zeta| \leq 1/2$, there exists a constant A_{μ,s,s_*} such that

$$|H_n(z) - 1|, |\log H_n(z)| \leq |1-z| A_{\mu,s,s_*}\big(s/s_*\big)^n, \quad s_* \leq |z| \leq 1/s_*.$$

Observe that the constants $A_{\mu,s,s_*} e^{\sqrt{n+1}}\big(s/s_*\big)^n$ are uniformly bounded above. Then the first claim of the lemma follows by minimizing these constants over all parameters $s < s_*$ between ρ and ρ_*. Further, it follows from Cauchy's formula that

$$H_n'(z) = \left(\int_{|\zeta|=1/s_*} - \int_{|\zeta|=s_*}\right) \frac{H_n(\zeta)-1}{(\zeta-z)^2}\frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}$$

for $\rho_* \leq |z| \leq 1/\rho_*$ and therefore it holds in this annulus that

$$|H_n'(z)| \leq C_{\mu,s,s_*,\rho_*}\big(s/s_*\big)^n.$$

The last claim of the lemma is now deduced in the same manner as the first one. $\square$

This conclude the auxiliary estimates section. Now we are prepared to introduce the proof of the main result.

3.3 Proof of Main Result

Using (1.13), it is easy to show that

$$\mathbb{E}_n(\mu) = \frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x.$$

Furthermore, if we define $\mathrm{d}\sigma(\xi) := \mu'(-\xi)|\mathrm{d}\xi|$, then $\sigma'(\xi) = \mu'(-\xi)$ is still holomorphic and positive on the unit circle. Moreover, $b_n(z;\sigma) = b_n(-z;\mu)$. Therefore,

$$\mathbb{E}_n(\mu) = \widehat{\mathbb{E}}_n(\mu) + \widehat{\mathbb{E}}_n(\sigma), \quad \widehat{\mathbb{E}}_n(\nu) := \frac{2}{\pi}\int_{0}^{1}\frac{\sqrt{1-h_{n+1}^2(x;\nu)}}{1-x^2}\mathrm{d}x, \tag{3.4}$$

for $\nu \in \{\mu, \sigma\}$. Thus, it is enough to investigate the asymptotic behavior of $\widehat{\mathbb{E}}_n(\mu)$. To this end, let

$$a := (n+1)^{1/2} \quad \text{and} \quad x =: 1 - t/(n+1), \quad 0 \le t \le a. \tag{3.5}$$

We shall also write

$$1 - h_{n+1}^2(x) =: f^2(t)(1 + E_n(t)), \tag{3.6}$$

for $1 - (n+1)^{-1/2} \le x \le 1$, where $f(t)$ was defined in (1.9).

Lemma 3.3.1. *Given an integer $N \ge 1$, it holds that*

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log(n+1) + \frac{1}{2}A_0 + G_n - \frac{1}{2}\sum_{p=1}^{N-1}H_p(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right)$$

for large n, where $\mathcal{O}_N(\cdot)$ is independent of n, but does depend on N,

$$G_n := \frac{1}{\pi}\int_0^a \left(t^{-1} + \big(2(n+1)-t\big)^{-1}\right) f(t)\left(\big(1+E_n(t)\big)^{1/2} - 1\right)\mathrm{d}t,$$

and $H_p := \dfrac{1}{2^{p-1}\pi}\displaystyle\int_0^\infty \big(1 - f(t)\big)t^{p-1}\mathrm{d}t$ *for $p \ge 1$.*

Proof. Set $\delta := 1 - (n+1)^{-1/2}$. It trivially holds that

$$\widehat{\mathbb{E}}_n(\mu) = \frac{2}{\pi}\int_0^\delta \frac{\mathrm{d}x}{1-x^2} - \frac{2}{\pi}\int_0^\delta \frac{1-\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x + \frac{2}{\pi}\int_\delta^1 \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x.$$

Denote the third integral above by B_n. The second integral above is positive and equals to

$$\frac{2}{\pi}\int_0^\delta \frac{h_{n+1}^2(x)}{1+\sqrt{1-h_{n+1}^2(x)}}\frac{\mathrm{d}x}{1-x^2} \le \frac{2}{\pi}\int_0^\delta h_{n+1}^2(x)\frac{\mathrm{d}x}{1-\delta^2} = \mathcal{O}\left(a^5e^{-2a}\right),$$

where we used Lemma 3.2.1 for the last estimate. Therefore,

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log\left(\frac{1+\delta}{1-\delta}\right) + B_n + o_N\left((n+1)^{-N}\right),$$

where $o_N(\cdot)$ is independent of n, but does depend on N. Substituting $x = 1-t/(n+1)$ into the expression for B_n and recalling (3.6), we get that

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_0^a f(t)\big(1+E_n(t)\big)^{1/2}\frac{2(n+1)}{t(2(n+1)-t)}\,\mathrm{d}t \\ &= \frac{1}{\pi}\left(\log 2 + \log\frac{1}{1+\delta}\right) + \frac{1}{\pi}\int_0^a \frac{f(t)}{t}\,\mathrm{d}t - \frac{1}{\pi}\int_0^a \frac{1-f(t)}{2(n+1)-t}\,\mathrm{d}t + G_n. \end{aligned}$$

It was shown in [6, Lemma 8] that

$$\frac{1}{\pi}\int_0^a \frac{1-f(t)}{2(n+1)-t}\,\mathrm{d}t = \frac{1}{2}\sum_{p=1}^{N-1} H_p(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

where $\mathcal{O}_N(\cdot)$ is independent of n, but does depend on N. Moreover, it holds that

$$\begin{aligned} &\frac{1}{\pi}\log\left(\frac{1+\delta}{1-\delta}\right) + \frac{1}{\pi}\left(\log 2+\log\frac{1}{1+\delta}\right) + \frac{1}{\pi}\int_0^a \frac{f(t)}{t}\,\mathrm{d}t = \\ &\qquad\qquad = \frac{1}{\pi}\log\frac{a}{1-\delta} + \frac{1}{2}A_0 + \frac{1}{\pi}\int_a^\infty \frac{1-f(t)}{t}\,\mathrm{d}t. \end{aligned}$$

Since $\log a - \log(1-\delta) = \log(n+1)$ and it was shown in [6, Lemma 7] that

$$\frac{1}{\pi}\int_a^\infty \frac{1-f(t)}{t}\,\mathrm{d}t = \mathcal{O}\left(ae^{-2a}\right) = o_N\left((n+1)^{-N}\right),$$

where as usual $o_N(\cdot)$ is independent of n, but does depend on N, the claim of the lemma follows. □

We continue by deriving a different representation for the functions $E_n(t)$. To this end, notice that $t^2\mathrm{csch}^2 t = 1 - t^2/3 + \mathcal{O}\big(t^4\big)$ as $t \to 0$ and therefore $f^2(t) = t^2/3 + \mathcal{O}\big(t^4\big)$ as $t \to 0$. Hence, the function

$$\chi(t) := \left(\frac{t^2\mathrm{csch} t}{f(t)}\right)^2 \tag{3.7}$$

is continuous and non-vanishing at zero. Once again, we use notation from (3.5).

Lemma 3.3.2. *Set* $b_{n+1}^2(x) =: e^{-\mu_n(t)-2t}$ *and* $b'_{n+1}(x) =: (n+1)e^{w_n(t)-t}$*. Then it holds that*

$$E_n(t) = t^{-2}\chi(t)\left[1 - \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{(1 + D_n(t))^2}\right], \quad D_n(t) := \frac{1 - e^{-\mu_n(t)}}{e^{2t} - 1}.$$

Moreover, $\lim_{t\to 0^+} E_n(t)$ *exists and is finite.*

Proof. Since $h_{n+1}(1) = 1$ and $x = 1 - t/(n+1)$, it follows from (3.6) and the L'Hôpital's rule that

$$\lim_{t\to 0^+} E_n(t) = \frac{6}{(n+1)^2} \lim_{x\to 1^-} \frac{1 - h_{n+1}(x)}{(1-x)^2} - 1 = \frac{3}{(n+1)^2} \lim_{x\to 1^-} \frac{h'_{n+1}(x)}{1-x} - 1.$$

Since $h_{n+1}(z)$ is a holomorphic function around 1, the latter limit is finite if and only if $h'_{n+1}(1) = 0$. As Blaschke products $b_{n+1}(z)$ satisfy $b_{n+1}(x)b_{n+1}(1/x) \equiv 1$, it holds that $h_{n+1}(x) = h_{n+1}(1/x)$, which immediately yields the desired equality.

To derive the claimed representation of $E_n(t)$, recall (1.13) and substitute $x = 1-t/(n+1)$ into (3.6) to get that

$$\begin{aligned}
f^2(t)(1 + E_n(t)) &= 1 - \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{4t^2 e^{2w_n(t)-2t}}{\big(1 - e^{-\mu_n(t)-2t}\big)^2} \\
&= 1 - \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{t^2\mathrm{csch}^2 t e^{2w_n(t)}}{\big(1 + D_n(t)\big)^2} \\
&= f^2(t)\left[1 + t^{-2}\chi(t)\left(1 - \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{\big(1 + D_n(t)\big)^2}\right)\right]
\end{aligned}$$

from which the first claim of the lemma easily follows. $\square$

In the next four lemmas we repeatedly use approximation by Taylor polynomials with the Lagrange remainder:

$$F(y) = \sum_{k=0}^{M-1} \frac{F^{(k)}(0)}{k!} y^K + \frac{F^{(M)}(\theta y)}{M!} y^M \tag{3.8}$$

for some $\theta \in (0,1)$ that dependents on both y and M.

Lemma 3.3.3. *Put $\omega(t) := t/(e^{2t}-1)$. Given an integer $N \geq 1$, it holds for all n large that*

$$\Big(1 + D_n(t)\Big)^{-2} = 1 + \sum_{p=1}^{N-1} \alpha_p(t)(n+1)^{-p} + \alpha_{n,N}(t)(n+1)^{-N},$$

where the functions $\alpha_p(t)$ are independent of n and N and are polynomials of degree p in ω with coefficients that are polynomials in t of degree at most $2p-1$, and the functions $\alpha_{n,N}(t)$ are bounded in absolute value for $0 \leq t \leq a$ by a polynomial of degree $2N-1$ whose coefficients are independent of n. Moreover,

$$\alpha_p(t) = (p+1)s_1^p - ps_1^{p-1}(2s_1+1)t + \mathcal{O}\Big(t^2\Big) \quad as \quad t \to 0.$$

Proof. We start by deriving an asymptotic expansion of $\mu_n(t)$. It follows from Lemma 3.2.3 that $\log H_n(x) = t\mathcal{O}(a^{-2}e^{-a}) = to_N(1)(n+1)^{-N}$ uniformly for $0 \leq t \leq a$. Fix T in Lemma 3.2.2 and let n_T be such that $1 < \sqrt{n_T+1}T$. Then it holds for all $n \geq n_T$ that

$$\log(SH_n)(x) = \sum_{p=1}^{N-1} c_p t^p (n+1)^{-p} + t\hat{c}_N(t)(n+1)^{-N},$$

where $|\hat{c}_N(t)| \leq C_{\mu,T,N} t^{N-1} + o_N(1)$ uniformly for $0 \leq t \leq a$ and $C_{\mu,T,N} \leq C_{\mu,T} T^{-N}$. Hence, it follows from (3.3) and [6, Lemma 2] that

$$\begin{aligned} \mu_n(t) &= -2(n+1)\log x - 2t - 2\log(SH_n)(x) \\ &= \sum_{p=1}^{N-1} t^p m_p(t)(n+1)^{-p} + t m_{n,N}(t)(n+1)^{-N}, \end{aligned} \tag{3.9}$$

where

$$m_p(t) := \Big(2(p+1)^{-1}t - 2c_p\Big) \quad \text{and} \quad m_{n,N}(t) := 2\hat{m}_{n,N}(t)t^N/(N+1) - 2\hat{c}_N(t)$$

with $1 \le \hat{m}_{n,N}(t) \le (3/2)^{N+1}$. Assuming that $T < 2/3$, we have that

$$|m_{n,N}(t)| \le C_{\mu,T,N}t^{N-1}(t+1) + o_N(1) \tag{3.10}$$

uniformly for $0 \le t \le a$ and $C_{\mu,T,N} \le C_{\mu,T}T^{-N}$. Using (3.9) with $N = 1$, we get that

$$|\mu_n(t)| = \left|\frac{tm_{n,1}(t)}{n+1}\right| \le \frac{|m_{n,1}(t)|}{\sqrt{n+1}} \le C_{\mu,T}, \quad 0 \le t \le a. \tag{3.11}$$

Recalling the definition of $D_n(t)$ in Lemma 3.3.2, we get from (3.8) that

$$D_n(t) = \omega(t)\frac{1-e^{-\mu_n(t)}}{t} = \omega(t)\left(-\frac{1}{t}\sum_{k=1}^{N-1}\frac{(-1)^k}{k!}\mu_n^k(t) - \frac{1}{t}e^{-\theta_1\mu_n(t)}\frac{(-1)^N}{N!}\mu_n^N(t)\right)$$

for some $\theta_1 \in (0,1)$ that depends on N and $\mu_n(t)$. Plugging (3.9) into the above formula gives us

$$D_n(t) = \omega(t)\sum_{p=1}^{N-1} t^{p-1}d_p(t)(n+1)^{-p} + \omega(t)d_{n,N}(t)(n+1)^{-N}, \tag{3.12}$$

where $d_p(t)$ is a polynomial of degree p with coefficients independent of n and N given by

$$d_p(t) := -\sum_{k=1}^{p}\frac{(-1)^k}{k!}\sum_{j_1+\cdots+j_k=p} m_{j_1}(t)\cdots m_{j_k}(t),$$

here, each index $j_i \in \{1,\ldots,p\}$, and $d_{n,N}(t)$ is given by

$$d_{n,N}(t) := -\sum_{k=1}^{N-1}\frac{(-1)^k}{k!}\sum_{j_1+\cdots+j_k\ge N}\frac{1}{t}\frac{m_{n,j_1,N}(t)\cdots m_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} - \frac{(-1)^N}{N!}\frac{(n+1)^N}{e^{\theta_1\mu_n(t)}}\frac{\mu_n^N(t)}{t}$$

with $m_{n,j,N}(t) := t^j m_j(t)$ when $j < N$ and $m_{n,N,N}(t) := tm_{n,N}(t)$. Recall that $t^2/(n+1) \le 1$ on $0 \le t \le a$ since $a = \sqrt{n+1}$. Hence, the first summand above is bounded in absolute value for $0 \leq t \leq a$ by a polynomial of degree $2N-1$ whose coefficients depend on N but are independent of n. We also get from (3.11) and (3.10) that

$$\left|e^{-\theta_1\mu_n(t)}(n+1)^N\mu_n^N(t)/t\right| \leq e^{C_{\mu,T}}t^{N-1}|m_{n,1}(t)|^N \leq C_{\mu,T}^* t^{N-1}(t+2)^N$$

for $0 \leq t \leq a$. Further, using (3.12) with $N=1$ and (3.11) gives us

$$|D_n(t)| = \frac{\omega(t)}{e^{\theta_1\mu_n(t)}}\left|\frac{\mu_n(t)}{t}\right| \leq \frac{e^{C_{\mu,T}}}{2}\frac{|m_{n,1}(t)|}{n+1} \leq \frac{C_{\mu,T}e^{C_{\mu,T}}}{2\sqrt{n+1}}, \quad 0 \leq t \leq a. \tag{3.13}$$

Notice also that since $c_1 = s_1$ and $c_2 = s_1/2$ by Lemma 3.2.2, we have that

$$d_1(t) = t - 2s_1 \quad \text{and} \quad d_2(t) = -(1/2)t^2 + t(2s_1 + 2/3) - s_1(2s_1+1).$$

It follows from (3.13) that for any $-1 < D < 0$, there exists an integer $n_D \geq n_T$ such that $D \leq D_n(t)$ for $0 \leq t \leq a$ and $n \geq n_D$. Hence, we get from (3.8) that

$$\big(1 + D_n(t)\big)^{-2} = 1 + \sum_{k=1}^{N-1}(-1)^k(k+1)D_n^k(t) + \frac{(-1)^N(N+1)D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}$$

for all $n \geq n_D$ and some $\theta_2 \in (0,1)$ that depends on N and $D_n(t)$. Then the statement of the lemma follows with

$$\alpha_p(t) := \sum_{k=1}^{p}(-1)^k(k+1)\omega^k(t)t^{p-k}\sum_{j_1+\cdots+j_k=p} d_{j_1}(t)\cdots d_{j_k}(t)$$

here again, each index $j_i \in \{1,\ldots,p\}$, and

$$\alpha_{n,N}(t) := \sum_{k=1}^{N-1}(-1)^k(k+1)\omega^k(t)\sum_{j_1+\cdots+j_k\geq N}\frac{d_{n,j_1,N}(t)\cdots d_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + (n+1)^N\frac{(-1)^N(N+1)D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}$$

with $d_{n,j,N}(t) := t^{j-1}d_j(t)$ when $j < N$ and $d_{n,N,N}(t) := d_{n,N}(t)$. Reasoning as before lets us conclude that the first summand in the definition of $\alpha_{n,N}(t)$ is bounded in absolute value for $0 \leq t \leq a$ by a polynomial of degree $2N-1$ whose coefficients depend on N but are independent of n. Moreover, since

$$\left|\frac{(n+1)^N D_n^N(t)}{(1+\theta_2 D_n(t))^{N+2}}\right| \leq \frac{e^{NC_{\mu,T}}|m_{n,1}(t)|^N}{2^N(1-D)^{N+2}} \leq \frac{C^*_{\mu,T}e^{NC_{\mu,T}}(t+2)^N}{2^N(1-D)^{N+2}}, \quad 0 \leq t \leq a,$$

by (3.13) and (3.10), the same is true for the second summand as well. Now, notice that

$$\alpha_p(t) = \Big(-\omega(t)d_1(t)\Big)^{p-2}\bigg((p+1)\Big(\omega(t)d_1(t)\Big)^2 - p(p-1)t\omega(t)d_2(t)\bigg) + \mathcal{O}\Big(t^2\Big)$$

as $t \to 0$. Since $2\omega(t) = 1 - t + \mathcal{O}\Big(t^2\Big)$ as $t \to 0$, the last claim of the lemma follows after a straightforward computation. $\square$

Lemma 3.3.4. *Given $N \geq 1$, it holds for all n large that*

$$e^{2w_n(t)} = 1 + \sum_{p=1}^{N-1}\beta_p(t)(n+1)^{-p} + \beta_{n,N}(t)(n+1)^{-N},$$

where $\beta_p(t)$ is a polynomial of degree $2p$ whose coefficients are independent of n and N and the functions $\beta_{n,N}(t)$ are bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N$ whose coefficients are independent of n. Moreover, as $t \to 0$, it holds that

$$\begin{cases} \beta_1(t) = -2s_1 + 2(s_1+1)t - t^2, \\ \beta_2(t) = s_1^2 - 4s_1(s_1+1)t + \mathcal{O}\Big(t^2\Big), \\ \beta_3(t) = 2s_1^2(s_1+1)t + \mathcal{O}\Big(t^2\Big), \\ \beta_p(t) = \mathcal{O}\Big(t^2\Big), \quad p \geq 4. \end{cases}$$

Proof. We start by deriving an asymptotic expansion for $w_n(t)$. It follows from the very definition of $w_n(t)$ in Lemma 3.3.2, (3.3), and [6, Lemma 2] that

$$\begin{aligned} w_n(t) &= t + \log\frac{b_{n+1}(x)}{n+1} = t + n\log x + \log\left((SH_n)(x) + \frac{x(SH_n)(x)}{n+1}\right) \\ &= \sum_{p=1}^{N-1} t^p\phi_p(t)(n+1)^{-p} + \phi_{n,N}(t)(n+1)^{-N} + \log\left((SH_n)(x) + \frac{x(SH_n)(x)}{n+1}\right), \end{aligned}$$

where

$$\phi_p(t) := \frac{p+1-pt}{p(p+1)} \quad \text{and} \quad \phi_{n,N}(t) := \left(N^{-1} - \frac{n\hat{m}_{n,N}(t)t}{(N+1)(n+1)}\right)t^N \tag{3.14}$$

with some $1 \le \hat{m}_{n,N}(t) \le (3/2)^N$. Further, notice that

$$(S^{(i)}H_n)(x) = S^{(i)}(x) + o_N(1)(n+1)^{-N} \quad \text{and} \quad (SH_n')(x) = o_N(1)(n+1)^{-N}$$

uniformly for $0 \le t \le a$, $i \in \{0,1\}$, by Lemma 3.2.3 and since $S(z)$ is a fixed holomorphic function in a neighborhood of 1. Fix T in Lemma 3.2.2. Then it holds for all $n \ge n_T$ that

$$(SH_n)(x) = 1 + \sum_{j=1}^{N-1} s_j \frac{t^j}{(n+1)^j} + \hat{s}_N(t)(n+1)^{-N},$$

and

$$(SH_n)(x) = -\sum_{j=1}^{N-1} js_j \frac{t^{j-1}}{(n+1)^{j-1}} - \hat{f}_N(t)(n+1)^{-N},$$

where $|\hat{s}_N(t)|, |\hat{f}_N(t)| \le C_\mu(t/T)^N + o_N(1)$ uniformly for $0 \le t \le a$. Therefore,

$$L_n(t) := (SH_n)(x) - 1 + \frac{x(SH_n)(x)}{n+1} = \sum_{j=1}^{N-1} t^{j-1} l_j(t)(n+1)^{-j} + l_{n,N}(t)(n+1)^{-N}, \tag{3.15}$$

where

$$l_j(t) := \Big(s_j(t-j) + (j-1)s_{j-1}\Big)$$

and

$$l_{n,N}(t) := (N-1)s_{N-1}t^{N-1} + \hat{s}_N(t) - \left(1 - \frac{t}{n+1}\right)\frac{\hat{f}_N(t)}{n+1}.$$

In particular, it holds that

$$|l_{n,N}(t)| \le 2C_\mu(t/T)^N + (N-1)s_{N-1}t^{N-1} + o_N(1) \tag{3.16}$$

and therefore

$$|L_n(t)| \le \frac{|l_{n,1}(t)|}{n+1} \le \frac{C_{\mu,T}}{\sqrt{n+1}}, \quad 0 \le t \le a. \tag{3.17}$$

Hence, given $-1 < L < 0$, there exists an integer $n_L \geq n_T$ such that $L \leq L_n(t)$ for $0 \leq t \leq a$ and $n \geq n_L$. Thus, we get from (3.8) that

$$\log(1 + L_n(t)) = \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k} L_n^k(t) + \frac{(-1)^{N-1} L_n^N(t)}{N(1 + \theta_3 L_n(t))^N}$$

for some $\theta_3 \in (0, 1)$ that depends on N and $L_n(t)$. Therefore, we get from (3.15) that

$$\log\left((SH_n)(x) + \frac{x(SH_n)(x)}{n+1}\right) = \sum_{p=1}^{N-1} \psi_p(t)(n+1)^{-p} + \psi_{n,N}(t)(n+1)^{-N},$$

where $\psi_p(t)$ is a polynomial of degree p with coefficients independent of n and N given by

$$\psi_p(t) := \sum_{k=1}^{p} \frac{(-1)^{k-1}}{k} \sum_{j_1+\cdots+j_k=p} t^{p-k} l_{j_1}(t) \cdots l_{j_k}(t), \tag{3.18}$$

here, each index $j_i \in \{1, \ldots, p\}$, and $\psi_{n,N}(t)$ is given by

$$\psi_{n,N}(t) := \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k} \sum_{j_1+\cdots+j_k \geq N} \frac{l_{n,j_1,N}(t) \cdots l_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + (n+1)^N \frac{(-1)^{N-1} L_n^N(t)}{N(1+\theta_3 L_n(t))^N}$$

with $l_{n,j,N}(t) := t^{j-1} l_j(t)$ when $j < N$ and $l_{n,N,N}(t) := l_{n,N}(t)$. As in the previous lemma, since $t^2/(n+1) \leq 1$ when $0 \leq t \leq a$, the first summand above is bounded in absolute value by a polynomial of degree N whose coefficients are independent of n. It also follows from (3.17) and (3.16) that

$$\frac{(n+1)^N |L_n^N(t)|}{|1 + \theta_3 L_n(t)|^N} \leq \frac{|l_{n,1}(t)|^N}{(1-L)^N} \leq C_{\mu,T} \frac{(t+1)^N}{(1-L)^N}, \quad 0 \leq t \leq a,$$

for all $n \geq n_L$. Altogether, we have shown that

$$w_n(t) = \sum_{p=1}^{N-1} \Big(t^p \phi_p(t) + \psi_p(t)\Big)(n+1)^{-p} + \Big(\phi_{n,N}(t) + \psi_{n,N}(t)\Big)(n+1)^{-N} \tag{3.19}$$

with ϕ_p, ψ_p and $\phi_{n,N}, \psi_{n,N}$ as described above. We also can deduce from (3.14) and (3.18) that $t\phi_1(t) + \psi_1(t) = -s_1 + t(s_1+1) - t^2/2$ and

$$t^p\phi_p(t) + \psi_p(t) = \frac{(-1)^{p-1}}{p} l_1^p(t) + (-1)^{p-2} t l_1^{p-2}(t) l_2(t) + \mathcal{O}\big(t^2\big) = -\frac{s_1^p}{p} + \mathcal{O}\big(t^2\big) \tag{3.20}$$

for $p \geq 2$, where we used that $2s_2 = s_1^2 + s_1$, see Lemma 3.2.2. Since

$$\Big|\psi_{n,1}(t)\Big| \leq (n+1)\frac{|L_n(t)|}{1-L} \leq \sqrt{n+1}\frac{C_{\mu,T}}{1-L}, \quad 0 \leq t \leq a,$$

by (3.17) for $n \geq n_L$, we get from (3.19), applied with $N = 1$, and (3.14) that

$$|w_n(t)| = \left|\frac{\phi_{n,1}(t) + \psi_{n,1}(t)}{n+1}\right| \leq C_{\mu,T,L}, \quad 0 \leq t \leq a, \quad n \geq n_L. \tag{3.21}$$

Now, using (3.8) once more, we get

$$e^{2w_n(t)} = 1 + \sum_{k=1}^{N-1} \frac{2^k}{k!} w_n^k(t) + e^{2\theta_4 w_n(t)} \frac{2^N}{N!} w_n^N(t)$$

for some $\theta_4 \in (0,1)$ that depends on N and $w_n(t)$. Plugging (3.19) into the above formula gives us the desired expansion with

$$\beta_p(t) := \sum_{k=1}^{p} \frac{2^k}{k!} \sum_{j_1+\cdots+j_k=p} \Big(t^{j_1}\phi_{j_1}(t) + \psi_{j_1}(t)\Big) \cdots \Big(t^{j_k}\phi_{j_k}(t) + \psi_{j_k}(t)\Big), \tag{3.22}$$

which is a polynomial of degree $2p$ with coefficients independent of n and N, and

$$\beta_{n,N}(t) := \sum_{k=1}^{N-1} \frac{2^k}{k!} \sum_{j_1+\cdots+j_k \geq N} \frac{\prod_{i=1}^k \Big(\phi_{n,j_i,N}(t) + \psi_{n,j_i,N}(t)\Big)}{(n+1)^{j_1+\cdots+j_k-N}} + e^{2\theta_4 w_n(t)} \frac{2^N}{N!}(n+1)^N w_n^N(t)$$

with $\phi_{n,j,N}(t) := t^j\phi_j(t), \psi_{n,j,N}(t) := \psi_j(t)$ when $j < N$ and $\phi_{n,N,N}(t) := \phi_{n,N}(t), \psi_{n,N,N}(t) := \psi_{n,N}(t)$, which is bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N$ whose coefficients are independent of n due to (3.21) and the same reasons as in the similar previous computations. Thus, it only remains to compute the linear approximation to $\beta_p(t)$ at zero. Now, it follows from (3.20) and (3.22) that

$$\beta_p(t) = s_1^p \sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{1}{j_1 \cdots j_k} - \left(s_1^{p-1}(s_1+1) \sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{n(j_1,\ldots,j_k)}{j_1 \cdots j_k} \right) t + \mathcal{O}\big(t^2\big)$$

where $n(j_1,\ldots,j_k)$ is the number of 1's in the partition $\{j_1,\ldots,j_k\}$ of p. To simplify this expression observe that

$$\begin{aligned}
(1-x)^2 e^{-2yx} = e^{2\log(1-x)-2yx} &= 1 + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} \big(yx - \ln(1-x) \big)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-2)^k}{k!} \left((1+y)x + \sum_{j=2}^{\infty} \frac{x^j}{j} \right)^k \\
&= 1 + \sum_{p=1}^{\infty} \left(\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{(1+y)^{n(j_1,\ldots,j_k)}}{j_1 \cdots j_k} \right) x^p,
\end{aligned} \tag{3.23}$$

where y is a free parameter. By putting $y=0$ in this expression, we get that

$$\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{1}{j_1 \cdots j_k} = \begin{cases} -2 & \text{if} \quad p=1, \\ 1 & \text{if} \quad p=2, \\ 0 & \text{if} \quad p \geq 3. \end{cases}$$

Moreover, by differentiating (3.23) with respect to y and then putting $y=0$, we get

$$\sum_{k=1}^{p} \frac{(-2)^k}{k!} \sum_{j_1+\cdots+j_k=p} \frac{n(j_1,\ldots,j_k)}{j_1 \cdots j_k} = \begin{cases} -2 & \text{if} \quad p=1, \\ 4 & \text{if} \quad p=2, \\ -2 & \text{if} \quad p=3, \\ 0 & \text{if} \quad p \geq 4, \end{cases}$$

which clearly finishes the proof of the last claim of the lemma. □

Lemma 3.3.5. *Let $\chi(t)$ be given by (3.7). For any integer $N \geq 1$, it holds that*

$$\Big(1 + E_n(t)\Big)^{1/2} - 1 = \chi(t) \sum_{p=1}^{N-1} u_p(t)(n+1)^{-p} + \chi(t) u_{n,N}(t)(n+1)^{-N},$$

where $u_p(t)$ is bounded in absolute value[1] *on $0 \leq t < \infty$ by a polynomial of degree $2p-2$ whose coefficients are independent of n and N and the functions $u_{n,N}(t)$ are bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N-2$ whose coefficients are independent of n.*

Proof. Set

$$R_n(t) := \left(1 - \frac{t}{2(n+1)}\right)^2 \frac{e^{2w_n(t)}}{(1 + D_n(t))^2}.$$

Lemmas 3.3.3 and 3.3.4 yield that $R_n(t)$ has the following asymptotic expansion:

$$R_n(t) = 1 + \sum_{p=1}^{N-1} r_p(t)(n+1)^{-p} + r_{n,N}(t)(n+1)^{-N},$$

where

$$r_p(t) := \sum_{j=0}^{p} \beta_j(t)\alpha_{p-j}(t) - \sum_{j=0}^{p-1} t\beta_j(t)\alpha_{p-1-j}(t) + \sum_{j=0}^{p-2} t^2 \beta_j(t)\alpha_{p-2-j}(t)/4$$

with $\alpha_0(t) = \beta_0(t) :\equiv 1$, and $r_{n,N}(t)$ given by

$$\sum_{k=N}^{2N+2} \left(\sum_{j=0}^{k} \frac{\beta_{n,j,N}(t)\alpha_{n,k-j,N}(t)}{(n+1)^{k-N}} - \sum_{j=0}^{k-1} \frac{t\beta_{n,j,N}(t)\alpha_{n,k-1-j,N}(t)}{(n+1)^{k-N}} + \sum_{j=0}^{k-2} \frac{t^2\beta_{n,j,N}(t)\alpha_{n,k-2-j,N}(t)/4}{(n+1)^{k-N}} \right)$$

with $\alpha_{n,j,N}(t) := \alpha_j(t), \beta_{n,j,N}(t) := \beta_j(t)$ when $j < N$, $\alpha_{n,N,N}(t) := \alpha_{n,N}(t), \beta_{n,N,N}(t) := \beta_{n,N}(t)$, and $\alpha_{n,j,N}(t) = \beta_{n,j,N}(t) :\equiv 0$ when $j > N$. It also follows from Lemmas 3.3.3 and 3.3.4 that the functions $r_p(t)$ are independent of n and N and are polynomials in ω of degree p with coefficients that are polynomials in t of degree at most $2p$, while the functions $r_{n,N}(t)$ are bounded in absolute value for $0 \leq t \leq a$ by a polynomial of degree $2N$ whose coefficients are independent of n. Finally, we get from Lemmas 3.3.3 and 3.3.4 that

$$\sum_{j=0}^{1} \beta_j(t)\alpha_{1-j}(t) = t + \mathcal{O}\Big(t^2\Big) \quad \text{and} \quad \sum_{j=0}^{k} \beta_j(t)\alpha_{k-j}(t) = \mathcal{O}\Big(t^2\Big)$$

[1]In fact, $u_p(t)$ is a multivariate polynomial in ω, χ, and t.

for all $k \geq 2$. Therefore, it holds that $r_p(t) = \mathcal{O}\Big(t^2\Big)$ as $t \to 0$ for all $p \geq 1$.

It follows from Lemma 3.3.2 that $E_n(t) = t^{-2}\chi(t)[1 - R_n(t)]$. Hence, plugging the expansion of $R_n(t)$ into this formula gives us

$$E_n(t) = \chi(t)\left[\sum_{p=1}^{N-1} e_p(t)(n+1)^{-p} + e_{n,N}(t)(n+1)^{-N}\right],$$

where $e_p(t) := -t^{-2}r_p(t)$ for any p and $e_{n,N}(t) := -t^{-2}r_{n,N}(t)$ for any n, N. It follows from the properties of $r_p(t)$ that each $e_p(t)$ is a continuous function and is bounded in absolute value on $0 \leq t < \infty$ by a polynomial of degree $2p-2$. Also, since $\chi(t)$ is a continuous function as well and $\lim_{t\to 0^+} E_n(t)$ exists and is finite according to Lemma 3.3.2, so must $\lim_{t\to 0^+} e_{n,N}(t)$ for all n, N. Then it follows from properties of $r_{n,N}(t)$ that $e_{n,N}(t)$ is bounded in absolute value when $0 \leq t \leq a$ by a polynomial of degree $2N-2$ whose coefficients are independent of n.

From what precedes, we get that

$$|E_n(t)| \leq \frac{\chi(t)|e_{n,1}(t)|}{n+1} \leq \frac{C_{\mu,T}}{n+1}, \quad 0 \leq t \leq a.$$

Hence, for any $-1 < E < 0$ there exists an integer n_E such that $E \leq E_n(t)$ for all $0 \leq t \leq a$ and $n \geq n_E$. Thus, by applying (3.8) one more time, we get that

$$(1+E_n(t))^{1/2} - 1 = \sum_{k=1}^{N-1}\binom{1/2}{k}E_n^k(t) + \binom{1/2}{N}\frac{E_n^N(t)}{(1+\theta_5 E_n(t))^{N-1/2}}$$

for some $\theta_5 \in (0,1)$ that depends on N and $E_n(t)$. Therefore, the claim of the lemma follows with

$$u_p(t) := \sum_{k=1}^{p}\binom{1/2}{k}\chi^{k-1}(t)\sum_{j_1+\cdots+j_k=p} e_{j_1}(t)\cdots e_{j_k}(t),$$

which is bounded in absolute value on $0 \leq t < \infty$ by a polynomial of degree $2p-2$ whose coefficients are independent of n and N, and

$$u_{n,N}(t) := \sum_{k=1}^{N-1}\binom{1/2}{k}\chi^{k-1}(t)\sum_{j_1+\cdots+j_k\geq N}\frac{e_{n,j_1,N}(t)\cdots e_{n,j_k,N}(t)}{(n+1)^{j_1+\cdots+j_k-N}} + \binom{1/2}{N}\frac{(n+1)^N E_n^N(t)}{(1+\theta_5 E_n(t))^{N-1/2}}$$

where $e_{n,j,N}(t) := e_j(t)$ when $j < N$ and $e_{n,N,N}(t) := e_{n,N}(t)$, which is bounded in absolute value on $0 \le t \le a$ by a polynomial of degree $2N-2$ whose coefficients are independent of n due to the same reasoning as in two previous lemmas. $\square$

Lemma 3.3.6. *Given $N \ge 1$, it holds that*

$$\frac{(1+E_n(t))^{1/2}-1}{2(n+1)-t} = \chi(t)\sum_{p=2}^{N-1} v_p(t)(n+1)^{-p} + \chi(t)v_{n,N}(t)(n+1)^{-N},$$

where $v_p(t)$ is bounded in absolute value on $0 \le t < \infty$ by a polynomial of degree $2p-4$ whose coefficients are independent of n and N and the functions $v_{n,N}(t)$ is bounded in absolute value when $0 \le t \le a$ by a polynomial of degree $2N-4$ whose coefficients are independent of n.

Proof. Since $0 \le t \le a = \sqrt{n+1}$, we get from (3.8) that

$$\frac{1}{2(n+1)-t} = \sum_{p=1}^{N-1} z_p(t)(n+1)^{-p} + z_{n,N}(t)(n+1)^{-N},$$

where

$$z_p(t) := 2^{-p}t^{p-1} \quad \text{and} \quad z_{n,N}(t) := \frac{2^{-N}t^{N-1}}{(1-\theta_6 t/2(n+1))^{N+1}}$$

for some $\theta_6 \in (0,1)$ that depends on N and t. Therefore, the claim of the lemma follows from Lemma 3.3.5 with

$$v_p(t) := \sum_{j=1}^{p-1} z_j(t)u_{p-j}(t) \quad \text{and} \quad v_{n,N}(t) := \sum_{k=N}^{2N}\sum_{j_1+j_2=k} \frac{z_{n,j_1,N}(t)v_{n,j_2,N}(t)}{(n+1)^{k-N}}$$

where $j_1, j_2 \in \{1,\dots,N\}$, $z_{n,j,N}(t) := z_j(t)$, $u_{n,j,N}(t) := u_j(t)$ for $j < N$, and $z_{n,n,N}(t) := z_{n,N}(t)$, $u_{n,N,N}(t) := u_{n,N}(t)$. $\square$

With the notation introduced in Lemmas 3.3.1, 3.3.5, and 3.3.6, the following lemma holds.

Lemma 3.3.7. *Given $N \ge 1$, it holds that*

$$G_n = I_1^\mu (n+1)^{-1} + \sum_{p=2}^{N-1}\left(I_p^\mu + J_p^\mu\right)(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right)$$

for all n large, where

$$I_p^\mu := \frac{1}{\pi}\int_0^\infty t^{-1} f(t)\chi(t)u_p(t)\mathrm{d}t \quad \textit{and} \quad J_p^\mu := \frac{1}{\pi}\int_0^\infty f(t)\chi(t)v_p(t)\mathrm{d}t$$

(observe that $t^{-1}f(t)$ is a continuous and bounded function on $0 \le t < \infty$, $\chi(t)$ decreases exponentially at infinity, and the functions $u_p(t), v_p(t)$ are bounded by polynomials).

Proof. By the very definition of G_n in Lemma 3.3.1 we have that $G_n = I_n + J_n$, where

$$I_n := \frac{1}{\pi}\int_0^a t^{-1} f(t)\left((1+E_n(t))^{1/2} - 1\right)\mathrm{d}t$$

and

$$J_n := \frac{1}{\pi}\int_0^a f(t)\frac{(1+E_n(t))^{1/2}-1}{2(n+1)-t}\mathrm{d}t.$$

Using Lemma 3.3.5, we can rewrite the first integral above as

$$I_n = \sum_{p=1}^{N-1} I_p^\mu (n+1)^{-p} - S_n + T_n,$$

where

$$S_n := \frac{1}{\pi}\sum_{p=1}^{N-1}(n+1)^{-p}\int_a^\infty t^{-1} f(t)\chi(t)u_p(t)\mathrm{d}t$$

and

$$T_n := \frac{1}{\pi}(n+1)^{-N}\int_0^a t^{-1} f(t)\chi(t)u_{n,N}(t)\mathrm{d}t.$$

Since $u_p(t) = \mathcal{O}\Big(t^{2p-2}\Big)$, $f(t) = \mathcal{O}(1)$, and $\chi(t) = \mathcal{O}\Big(t^4 e^{-2t}\Big)$ as $t \to \infty$, it holds that

$$\begin{aligned} S_n = \sum_{p=1}^{N-1}(n+1)^{-p}\int_a^\infty \mathcal{O}\Big(t^{2p+1}e^{-2t}\Big)\mathrm{d}t = \sum_{p=1}^{N-1}(n+1)^{-p}\mathcal{O}\Big(a^{2p+1}e^{-2a}\Big) = \\ = \mathcal{O}_N\left(ae^{-2a}\right) = o_N\left((n+1)^{-N}\right). \end{aligned}$$

Moreover, since $u_{n,N}(t)$ is bounded by a polynomial of degree $2N-2$ for $0 \le t \le a$, we have that $T_n = \mathcal{O}_N\Big((n+1)^{-N}\Big)$.

Similarly, we get from Lemma 3.3.6 that

$$J_n = \sum_{p=2}^{N-1} J_p^{\mu}(n+1)^{-p} - U_n + V_n,$$

where

$$U_n := \frac{1}{\pi}\sum_{p=2}^{N-1}(n+1)^{-p}\int_a^{\infty} f(t)\chi(t)v_p(t)\mathrm{d}t$$

and

$$V_n := \frac{1}{\pi}(n+1)^{-N}\int_0^a f(t)\chi(t)v_{n,N}(t)\mathrm{d}t.$$

An argument as above shows that $U_n = \mathcal{O}_N\left(e^{-2a}\right) = o_N\left((n+1)^{-N}\right)$ and $V_n = \mathcal{O}_N\left((n+1)^{-N}\right)$ for large n, which finishes the proof of the lemma. $\square$

Lemma 3.3.8. *The claim of Theorem 3.1.1 holds.*

Proof. It follows from Lemmas 3.3.1 and 3.3.7 that given an integer $N \geq 1$, it holds that

$$\widehat{\mathbb{E}}_n(\mu) = \frac{1}{\pi}\log(n+1) + \frac{1}{2}A_0 + \sum_{p=1}^{N-1}\left(I_p^{\mu} + J_p^{\mu} - H_p/2\right)(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

where we set $J_1^{\mu} := 0$. The claim of Theorem 3.1.1 now follows from (3.4) by taking $A_p^{\mu} := I_p^{\mu} + I_p^{\sigma} + J_p^{\mu} + J_p^{\sigma} - H_p$. $\square$

4. AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF KAC-GERONIMUS POLYNOMIALS

Let $\{\varphi_i(z;\alpha)\}_{i=0}^{\infty}$, corresponding to $\alpha \in (-1,1)$, be orthonormal Geronimus polynomials. We study asymptotic behavior of the expected number of real zeros, say $\mathbb{E}_n(\alpha)$, of random polynomials

$$P_n(z) := \sum_{i=0}^{n} \eta_i \varphi_i(z;\alpha),$$

where $\eta_0, \ldots, \eta_n$ are i.i.d. standard Gaussian random variables. When $\alpha = 0$, $\varphi_i(z;0) = z^i$ and $P_n(z)$ are called Kac polynomials. In this case it was shown by Wilkins that $\mathbb{E}_n(0)$ admits an asymptotic expansion (1.8) Below, we obtain a similar expansion of $\mathbb{E}(\alpha)$ for $\alpha \neq 0$.

4.1 Introduction and Main Results

In this chapter we continue investigating the asymptotic behavior of the expected number of real zeros of random polynomials (1.12). The previous results suggest that the constant $\pi/2$ in front of $\log(n+1)$ in (1.7) and (1.8) might change if the recurrence coefficients decay slowly or do not decay at all. Below, we support this guess by considering random polynomials of the form

$$P_n(z) = \eta_0\varphi_0(z;\alpha) + \eta_1\varphi_1(z;\alpha) + \cdots + \eta_n\varphi_n(z;\alpha), \tag{4.1}$$

that we call Kac-Geronimus polynomials, where η_i are i.i.d. standard real Gaussian random variables and

$$\varphi_m(z;\alpha) = \rho^{-m}\Phi_m(z;\alpha), \quad \rho := \sqrt{1-\alpha^2}, \tag{4.2}$$

are real Geronimus polynomials, that is, polynomials $\Phi_m(z;\alpha)$ satisfy (1.2) with $\alpha_m = \alpha \in (-1,1)$ for all $m \geq 0$. The measure of orthogonality for general Geronimus polynomials, i.e., $\alpha_m = \alpha \in \mathbb{D}$, is explicitly known, see [3, Section 1.6], and is supported by

$$\Delta_\alpha := \left\{ e^{i\theta} : 2\arcsin(|\alpha|) \leq \theta \leq 2\pi - 2\arcsin(|\alpha|) \right\}$$

with a possible pure mass point, which is present if and only if $|\alpha+1/2|>1/2$. When $\alpha=0$, one can clearly see from (1.2) that $\Phi_m(z;0)=z^m$ and therefore Kac-Geronimus polynomials (4.1) specialize to Kac polynomials (1.5). Recall (1.13).

Theorem 4.1.1 (Auxiliary Estimates)**.** *Let $P_n(z)$ be given by (4.1)–(4.2) with $\alpha\in(-1,0)\cup(0,1)$. Define*

$$r(z):=\sqrt{(z-1)^2+4\alpha^2 z} \tag{4.3}$$

to be the branch holomorphic in $\mathbb{C}\backslash\Delta_\alpha$ such that $r(z)/z\to 1$ as $z\to\infty$. Then it holds that

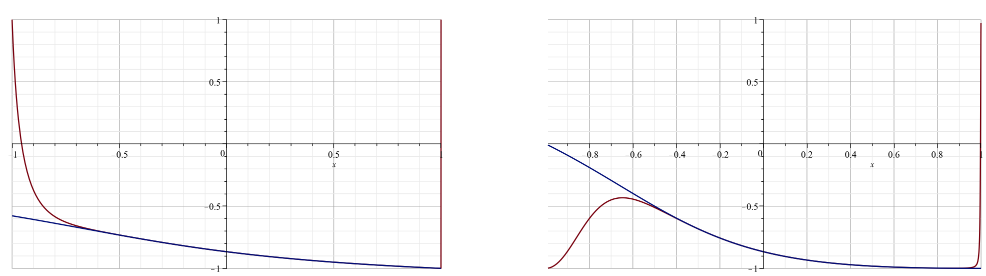
$$\lim_{n\to\infty} b_{n+1}(z)=\frac{-2\alpha}{r(z)+1-z} \tag{4.4}$$

locally uniformly in $\mathbb{D}$. Moreover, it holds that

$$h_{n+1}(x)=-\alpha\frac{x+1}{r(x)}\left(1+\mathcal{O}\left((1-x)^2(n+1)e^{-\sqrt{n+1}/\rho}\right)\right), \tag{4.5}$$

for $-1+(n+1)^{-1/2}\le x\le 1-\delta_\alpha^{n+1}$, where $\mathcal{O}(\cdot)$ does not depend on n and $\delta_\alpha:=0$ when $\alpha<0$ while $\delta_\alpha:=((1-\alpha)/(1+\alpha))^{1/3}$ when $\alpha>0$.

Observe that $b_{n+1}(1)=h_{n+1}(1)=1$ for all n and these equalities remain true in the limit when $\alpha<0$. However, $b(1)=h(1)=-1$ when $\alpha>0$. This change is due to a single zero of $\varphi_m(z;\alpha)$ that approaches 1 as $m\to\infty$ for every fixed $\alpha>0$, see Figure 4.1, and is the reason we need to introduce δ_α in (4.5).



((a)) $b_4(x)$ **and** $b(x)$ ((b)) $h_4(x)$ **and** $h(x)$

Figure 4.1. The graphs of $b_4(x)$ and $b(x)$ (panel (a)) and $h_4(x)$ and $h(x)$ (panel (b)) on $[-1,1]$ for $\alpha=\sqrt{3}/2$.

Let $\mathbb{E}_n(\alpha)$ be the expected number of real zeros of random polynomials (4.1)–(4.2). It is easy to see that $b_m(1/x) = 1/b_m(x)$ and therefore $b'_m(1/x) = x^2 b'_m(x)/b_m^2(x)$. Thus, we get from (1.13) that $h_m(1/x) = h_m(x)$ and therefore

$$\mathbb{E}_n(\alpha) = \frac{2}{\pi}\int_{-1}^{1} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x. \tag{4.6}$$

Using this formula we can prove the following theorem that constitutes the main result of this chapter.

Theorem 4.1.2 (Main result)**.** *Let $P_n(z)$ be random polynomials given by* (4.1)–(4.2) *with $\alpha \in (-1,0)\cup(0,1)$. Then there exist constants $A_p^{\alpha,(-1)^n}$, $p \geq 1$, that do depend on the parity of n, such that $\mathbb{E}_n(\alpha)$, the expected number of real zeros of $P_n(z)$, satisfies*

$$\mathbb{E}_n(\alpha) = \frac{1}{\pi}\log(n+1) + A_0^\alpha + \sum_{p=1}^{N-1} A_p^{\alpha,(-1)^n}(n+1)^{-p} + \mathcal{O}_N\left((n+1)^{-N}\right)$$

for any integer N, all n large, where $\mathcal{O}_N(\cdot)$ depends on N, but is independent of n, and

$$A_0^\alpha = \frac{A_0 + 1 + \operatorname{sgn}(\alpha)}{2} + \frac{1}{\pi}\log\frac{2}{|\alpha|}$$

with A_0 given by (1.9) *and* $\operatorname{sgn}(\alpha) := \alpha/|\alpha|$.

Notice that $A_0^{|\alpha|} = A_0^{-|\alpha|} + 1$. This is due to the fact that polynomials $\varphi_m(x;|\alpha|)$ have a zero exponentially close to 1 while polynomials $\varphi_m(x;-|\alpha|)$ do not have such a zero.

4.2 Proof of the Auxiliary Estimates

Lemma 4.2.1. *It holds that*

$$b_{n+1}(z) = \frac{\phi(z) - 2(1+\alpha) - \epsilon^{n+1}(z)(\psi(z) - 2(1+\alpha))}{\phi(z) - 2(1+\alpha)z - \epsilon^{n+1}(z)(\psi(z) - 2(1+\alpha)z)} \tag{4.7}$$

where $\phi(z) := z+1+r(z)$, $\psi(z) := z+1-r(z)$, $\epsilon(z) := \psi(z)/\phi(z)$, and $r(z)$ was defined in (4.3)*. In particular,* (4.4) *takes place.*

Proof. Let $U_m(y)$ be the degree m Chebyshëv polynomial of the second kind, that is,

$$U_m(y) = \frac{\left(y + \sqrt{y^2-1}\right)^{m+1} - \left(y - \sqrt{y^2-1}\right)^{m+1}}{2\sqrt{y^2-1}},$$

where for definiteness we take the branch $\sqrt{y^2-1} = y + \mathcal{O}(1)$ as $y \to \infty$ with the cut along $[-1,1]$. It has been shown in [43, Theorem 3.1] that

$$\begin{cases} \varphi_m(z;\alpha) = z^{m/2}\left(U_m\left(\dfrac{z+1}{2\rho\sqrt{z}}\right) - \dfrac{1+\overline{\alpha}}{\rho\sqrt{z}}U_{m-1}\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)\right), \\ \varphi_m^*(z;\alpha) = z^{m/2}\left(U_m\left(\dfrac{z+1}{2\rho\sqrt{z}}\right) - \dfrac{\sqrt{z}(1+\alpha)}{\rho}U_{m-1}\left(\dfrac{z+1}{2\rho\sqrt{z}}\right)\right), \end{cases} \tag{4.8}$$

where $U_{-1}(y) \equiv 0$ and we take the branch $\sqrt{z}$ that is positive for positive reals (of course, in our case $\overline{\alpha} = \alpha$). Observe that the map

$$y(z) = (z+1)/(2\rho\sqrt{z})$$

takes $\mathbb{D}$ into $\{\mathrm{Re}(z) > 0\}\backslash[0,1/\rho]$, the right half-plane with the real segment $[0,1/\rho]$ removed, and its boundary values on Δ_α cover the real interval $[0,1]$ twice. Therefore,

$$\sqrt{y(z)^2-1} = r(z)/(2\rho\sqrt{z}), \quad z \in \mathbb{D}.$$

In particular, it follows from (4.8) that (4.7) holds. Observe that

$$|\epsilon(z)| = \left|\frac{y-\sqrt{y^2-1}}{y+\sqrt{y^2-1}}\right| = \left|y + \sqrt{y^2-1}\right|^{-2} < 1 \tag{4.9}$$

for $|z| < 1$. Hence, $b_{n+1}(z)$ converges pointwise and therefore locally uniformly ($|b_{n+1}(z)| < 1$ for $z \in \mathbb{D}$) to

$$\frac{z-(1+2\alpha)+r(z)}{1-(1+2\alpha)z+r(z)} = \frac{z-(1+2\alpha)+r(z)}{1-(1+2\alpha)z+r(z)}\frac{z-(1+2\alpha)-r(z)}{z-(1+2\alpha)-r(z)} = \frac{-2\alpha}{r(z)+1-z}. \qquad \square$$

Lemma 4.2.2. *Let $h(x) := -\alpha(x+1)/r(x)$. It holds that*

$$h_{n+1}(x) = h(x)\left(1 - \epsilon^{n+1}(x)\frac{\frac{n+1}{\alpha}\frac{(1-x)^2}{x}r(x) + 2R(x)(1-\epsilon^{n+1}(x))}{(1-\epsilon^{n+1}(x))(S(x)+R(x)\epsilon^{n+1}(x))}\right), \tag{4.10}$$

where $R(x) := r(x) + \alpha(1+x)$ and $S(x) := r(x) - \alpha(1+x)$.

Proof. It follows from (4.7) that

$$b_{n+1}(x) = 1 - \lambda\frac{(1-x)(1-\epsilon^{n+1}(x))}{D(x)},$$

where $\lambda := 2(1+\alpha)$ and $D(x) := \phi(x) - \lambda x - \epsilon^{n+1}(x)(\psi(x) - \lambda x)$. It can be readily checked that

$$1 - b_{n+1}^2(x) = 2\lambda\frac{(1-x)(1-\epsilon^{n+1}(x))(S(x)+R(x)\epsilon^{n+1}(x))}{D^2(x)}.$$

Observe that

$$D'(x) = \phi'(x) - \lambda - (n+1)\epsilon^n(x)\epsilon'(x)(\psi(x) - \lambda x) - \epsilon^{n+1}(x)(\psi'(x) - \lambda).$$

It further holds that

$$\begin{aligned} b'_{n+1}(x) &= \lambda\frac{D(x)(1-\epsilon^{n+1}(x) + (n+1)(1-x)\epsilon^n(x)\epsilon'(x)) + D'(x)(1-x)(1-\epsilon^{n+1}(x))}{D^2(x)} \\ &=: \lambda\frac{N_1(x) + (n+1)(1-x)\epsilon^n(x)\epsilon'(x)N_2(x) + N_3(x)\epsilon^{n+1}(x) + N_4(x)\epsilon^{2(n+1)}(x)}{D^2(x)}, \end{aligned}$$

where $N_3(x), N_4(x)$ do not contain terms with $\epsilon'(x)$. We have that

$$\begin{aligned} N_1(x) &= \phi(x) - \lambda x + (1-x)(\phi'(x) - \lambda) = -2\alpha + r(x) + r'(x)(1-x) \\ &= -2\alpha + 2\alpha^2\frac{1+x}{r(x)} = -2\alpha\frac{S(x)}{r(x)}. \end{aligned}$$

Furthermore, we have that

$$N_2(x) = D(x) - (\psi(x) - \lambda x)(1 - \epsilon^{n+1}(x)) = 2r(x) = R(x) + S(x).$$

It also holds that

$$N_3(x) = -(\phi(x) - \lambda x) - (\psi(x) - \lambda x) - (1-x)(\psi'(x) - \lambda + \phi'(x) - \lambda) = 4\alpha.$$

Finally, similarly to $N_1(x)$, we have that

$$N_4(x) = \psi(x) - \lambda x + (1-x)(\psi'(x) - \lambda) = -2\alpha\Big(R(x)/r(x)\Big).$$

Since

$$\epsilon'(x) = \Big((1-x)/x\Big)\Big(\epsilon(x)/r(x)\Big), \tag{4.11}$$

it follows from (1.13) that

$$h_{n+1}(x) = h(x)\frac{(1-\epsilon^{n+1}(x))(S(x) - R(x)\epsilon^{n+1}(x)) - \frac{n+1}{\alpha}\frac{(1-x)^2}{x}r(x)\epsilon^{n+1}(x)}{(1-\epsilon^{n+1}(x))(S(x) + R(x)\epsilon^{n+1}(x))}$$

from which the desired claim easily follows. □

Lemma 4.2.3. *Formula (4.5) takes place.*

Proof. It can be readily checked that the function $|y + \sqrt{y^2-1}|$ is an increasing function of t for $y = t$, $t \in [1, \infty)$ and $y = \pm it$, $t \in [0, \infty)$. Since $\epsilon(1) = (1-|\alpha|)/(1+|\alpha|)$, it therefore holds that

$$\begin{aligned}\max_{x\in[-1+(n+1)^{-1/2},1]} |\epsilon(x)|^n &= \left|\epsilon(-1+(n+1)^{-1/2})\right|^n \\ &= \left(1-(n+1)^{-1/2}/\rho + \mathcal{O}\left((n+1)^{-1}\right)\right)^n \leq C_1 e^{-\sqrt{n+1}/\rho}\end{aligned} \tag{4.12}$$

for some absolute constant $C_1 > 0$.

Assume that $\alpha < 0$. Then $|S(x)| \geq r(x) \geq 2|\alpha|\rho$ for $x \in [-1, 1]$. Also, since $|h(x)|$ is an increasing function on $[-1, 1]$, we have that $|h(x)| \leq 1$ for $x \in [-1, 1]$. Thus, we get from (4.10) and (4.12) that

$$\begin{aligned}|h_{n+1}(x) - h(x)| &\leq C_2(n+1)e^{-\sqrt{n+1}/\rho}\left((1-x)^2 + |R(x)|\right) \\ &\leq C_3(1-x)^2(n+1)e^{-\sqrt{n+1}/\rho}\end{aligned} \tag{4.13}$$

for some absolute constants C_2, C_3, where one needs to observe that $\epsilon(0) = 0$ and

$$S(x)R(x) = \rho^2(1-x)^2. \tag{4.14}$$

This proves the lemma in the case $\alpha < 0$.

Suppose that $\alpha > 0$. It is quite easy to see that estimate (4.13) remains valid on $[-1 + (n+1)^{-1/2}, 0]$. Observe also that $\epsilon(x) > 0$ and is increasing for $x \in (0,1]$, see (4.11), and $0 < R(x) < 4$ on $[-1,1]$. Then by using (4.14) again, we get that

$$\begin{aligned}(1-\epsilon^{n+1}(x))(S(x) + R(x)\epsilon^{n+1}(x)) \quad &\geq \quad S(x) - R(x)\epsilon^{2(n+1)}(x) \\ &\geq \quad (\rho^2/4)(1-x)^2 - 4\epsilon^{2(n+1)}(1)\end{aligned}$$

for $x \in [0,1]$. Notice $\delta_\alpha = \epsilon^{1/3}(1)$. Then

$$(\rho^2/4)(1-x)^2 - 4\epsilon^{2(n+1)}(1) > (\rho^2/8)\delta_\alpha^{2(n+1)}$$

for $x \in \left[0, 1 - \delta_\alpha^{(n+1)}\right]$ and n sufficiently large. Therefore, similarly to (4.13), it again follows from (4.14) that there exist a constant C_4 such that

$$|h_{n+1}(x) - h(x)| \leq C_4(1-x)^2(n+1)\left(\epsilon(1)/\delta_\alpha^2\right)^{n+1} = C_4(1-x)^2(n+1)\epsilon^{2(n+1)/3}(1)$$

for $x \in \left[0, 1 - \delta_\alpha^{(n+1)}\right]$. Since $\epsilon(1) < 1$, the desired estimates follows. $\square$

4.3 Proof of the Main Result

To prove Theorem 4.1.2 we shall use the following straightforward facts. If $F(y)$ is analytic around the origin, then

$$F\left(\frac{t}{n+1}\right) = \sum_{p=0}^{N-1} \frac{F_p t^p}{(n+1)^p} + \frac{\widetilde{F}_N(t)t^N}{(n+1)^N}, \quad \left|\widetilde{F}_N(t)\right| \leq C_F^{N+1}, \tag{4.15}$$

for $t \in I_n := \left[0, \sqrt{n+1}\right]$ and all $n \geq n_F$, where $F_p = F^{(p)}(0)/p!$, the last estimate follows from the extended Cauchy integral formula, and C_F is independent of n, N. Further, if functions $u(t), v(t)$ satisfy

$$g(t) = \sum_{p=0}^{N-1} \frac{B_p(g;t)}{(n+1)^p} + \frac{\widetilde{B}_N(g;t)}{(n+1)^N}, \tag{4.16}$$

with $g \in \{u, v\}$, then so does their product and

$$B_p(uv;t) = \sum_{k=0}^{p} B_k(u;t) B_{p-k}(v;t) \tag{4.17}$$

for $p \leq N-1$, while

$$\widetilde{B}_N(uv;t) = \sum_{l=0}^{N} \frac{1}{(n+1)^l} \sum_{k+m=N+l,\ k,m\leq N} B_{N,k}(u;t) B_{N,m}(v;t) \tag{4.18}$$

with $B_{N,k}(g;t) = B_k(g;t)$ for $k < N$ and $B_{N,N}(t) = \widetilde{B}_N(g;t)$. Finally, let $F(y)$ be as in (4.15) and $g(t)$ be as in (4.16) with $B_0(g;t) = 0$. Assume that the values of $g(t)$ lie the domain of holomorphy of $F(y)$ for all $n \geq n_g$. Then

$$F(g(t)) = F(0) + \sum_{p=1}^{N-1} \frac{B_p(F \circ g;t)}{(n+1)^p} + \frac{\widetilde{B}_N(F \circ g;t)}{(n+1)^N}, \tag{4.19}$$

with

$$B_p(F \circ g;t) = \sum \frac{F^{(m)}(0)}{m_1! \cdots m_{N-1}!} \prod_{k=1}^{N-1} B_k^{m_k}(g;t), \tag{4.20}$$

where $m = m_1 + \cdots + m_{N-1}$ and the sum is taken over all partitions $p = \sum_{i=1}^{N-1} i m_i$, $m_i \geq 0$, and

$$\widetilde{B}_N(F \circ g;t) = \sum_{l=0}^{N(N-1)} \frac{1}{(n+1)^l} \sum \frac{F^{(m)}(0)}{m_1! \cdots m_N!} \prod_{k=1}^{N} B_{N,k}^{m_k}(g;t), \tag{4.21}$$

where $m = m_1 + \cdots + m_N$, the inner sum is taken over all partitions $l + N = \sum_{i=1}^{N} i m_i$, $m_i \geq 0$, and $B_{N,k}(g;t)$ has the same meaning as in (4.18).

Lemma 4.3.1. *Let $t \in I_n = \left[0, \sqrt{n+1}\right]$. Then it holds for all $N \geq 1$ that*

$$r\left(-1+\frac{t}{n+1}\right) = 2\rho\left(\sum_{p=0}^{N-1}\frac{r_p t^p}{(n+1)^p} + \frac{\tilde{r}_N(t)t^N}{(n+1)^N}\right) \tag{4.22}$$

for some constants r_p and functions $\tilde{r}_N(t)$ that obey estimate in (4.15). In particular, $r_0 = 1$, $r_1 = -1/2$, $r_2 = (1-\rho^2)/(8\rho^2)$. Moreover, for $\epsilon(z)$, defined in Lemma 4.2.1, it holds that

$$\epsilon^{n+1}\left(-1+\frac{t}{n+1}\right) = (-1)^{n+1}e^{-t/\rho}\left(1+\sum_{p=1}^{N-1}\frac{t^{p+1}e_p(t)}{(n+1)^p} + \frac{t^{N+1}\tilde{e}_N(t)}{(n+1)^N}\right), \tag{4.23}$$

where $e_p(t)$ is a polynomial of degree $p-1$ independent of n, N, in particular, $e_1(t) \equiv -1/(2\rho)$, and $|\tilde{e}_N(t)|$ is bounded above on I_n by a polynomial of degree $N-1$ whose coefficients depend only on N.

Proof. Observe that for $y > 0$ it follows from (4.3) and the choice of the branch of $r(z)$ that

$$r(-1+y) = 2\rho\sqrt{1-y+y^2/(4\rho^2)},$$

where the root in right-hand side of the above equality is principal. Since the right-hand side above is analytic around the origin, expansion (4.22) follows from (4.15). An absolutely analogous argument yields the expansion

$$\log\left(-\epsilon\left(-1+\frac{t}{n+1}\right)\right) = \sum_{p=1}^{N}\frac{\epsilon_p t^p}{(n+1)^p} + \frac{\tilde{\epsilon}_{N+1}(t)t^{N+1}}{(n+1)^{N+1}}, \quad \epsilon_1 = -\frac{1}{\rho}, \quad \epsilon_2 = -\frac{1}{2\rho},$$

where $|\tilde{\epsilon}_{N+1}(t)|$ has an upper bound as in (4.15). Since we can write

$$\epsilon^{n+1}\left(-1+\frac{t}{n+1}\right) = \frac{(-1)^{n+1}}{e^{t/\rho}}\exp\left\{(n+1)\left(\log\left(-\epsilon\left(-1+\frac{t}{n+1}\right)\right) + \frac{1}{\rho}\frac{t}{n+1}\right)\right\},$$

it follows from (4.19)–(4.21) that (4.23) holds, where $e_p(t)$ is a polynomial of degree $p-1$ independent of n, N (notice that always $m \leq p$ in (4.20)) and $|\tilde{e}_N(t)|$ is bounded above on I_n by a polynomial of degree $N-1$ whose coefficients depend only on N (again, we use that $m \leq l+N$ in (4.21) and that $t^{2l} \leq (n+1)^l$ on I_n). □

Lemma 4.3.2. *Set $\gamma(s) := 2s/(e^s - e^{-s})$ and let $x = -1 + t/(n+1)$, $t \in I_n$. It holds that*

$$h_{n+1}(x) = h(x) - (-1)^{n+1}\frac{(1-x)^2}{4}\gamma(t/\rho)(1 + \Gamma_{n+1}(t)) \tag{4.24}$$

with $\Gamma_{n+1}(t)$ having an expansion of the form

$$\Gamma_{n+1}(t) = \sum_{p=1}^{N-1}\frac{H_p(t)}{(n+1)^p} + \frac{\widetilde{H}_N(t)}{(n+1)^N}, \tag{4.25}$$

for any $N \geq 2$, where $H_1(t) = t - (-1)^{n+1}(\alpha/2\rho)t + \mathcal{O}(t^2)$, $H_p(t) = \mathcal{O}(t^2)$, $p \geq 2$, and $\widetilde{H}_N(t) = \mathcal{O}(t^2)$ as $t \to 0$, $|H_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n, N, while $|\widetilde{H}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n.

Proof. Recall (4.10). Notice that

$$\Big(1 - \epsilon^{n+1}(x)\Big)\Big(S(x) + R(x)\epsilon^{n+1}(x)\Big) = S(x) + 2\alpha(x+1)\epsilon^{n+1}(x) - R(x)\epsilon^{2(n+1)}(x). \tag{4.26}$$

It follows from (4.22) that $S(x)$ and $R(x)$ have expansions as in (4.16) with

$$B_p(S;t) = B_p(R;t) = 2\rho r_p t^p, \quad p \neq 1, \quad B_1(S;t) = -(\alpha+\rho)t, \quad B_1(R;t) = (\alpha-\rho)t,$$

and $\widetilde{B}_N(S;t) = \widetilde{B}_N(R;t) = 2\rho\tilde{r}_N(t)$ for any $N \geq 2$. Therefore, we get from (4.17)–(4.18) and (4.23) that

$$R(x)\epsilon^{2(n+1)}(x) = 2\rho e^{-2t/\rho}\left(1 + \sum_{p=1}^{N-1}\frac{C_p(t)t^p}{(n+1)^p} + \frac{\widetilde{C}_N(t)t^N}{(n+1)^N}\right)$$

for any $N \geq 2$, where $C_1(t) = (\alpha - \rho - 2t)/(2\rho)$, $C_p(t) = r_p + tq_p(t)$ for some polynomial $q_p(t)$ of degree $p-1$ when $p \geq 2$, and $|\widetilde{C}_N(t)|$ is bounded above on I_n by a polynomial of degree N independent of n. Consequently, we get that the expression in (4.26) has an expansion

$$2\rho\Big(1 - e^{-2t/\rho}\Big)\left(1 + \frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1}\frac{D_p(t)t^p}{(n+1)^p} + \frac{\widetilde{D}_N(t)t^N}{(n+1)^N}\right)$$

for all $N \geq 2$, where

$$\begin{aligned} D_1(t) &= -\alpha \left(\frac{1-(-1)^{n+1}e^{-t/\rho}}{2} \right)^2 \frac{2t/\rho}{1-e^{-2t/\rho}} + \frac{t}{2} \left(\frac{2t/\rho}{e^{2t/\rho}-1} - 1 \right) \\ &= -\alpha \frac{1-(-1)^{n+1}}{2} + \mathcal{O}\left(t^2\right) \quad \text{as} \quad t \to 0, \end{aligned}$$

with $|D_1(t)|$ bounded above by a linear function independent of n, N, and

$$D_p(t) = r_p + \gamma(t/\rho) \left((-1)^{n+1} \alpha e_{p-1}(t) - \rho e^{-t/\rho} q_p(t) \right) /2$$

for all $p \geq 2$, with $|D_p(t)|$ being bounded above on $[0, \infty)$, and $|\widetilde{D}_N(t)|$ that is bounded on I_n by a constant that depends on N but not on n. In particular, we have that

$$\left| \frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1} \frac{D_p(t) t^p}{(n+1)^p} + \frac{\widetilde{D}_N(t) t^N}{(n+1)^N} \right| = \left| \frac{D_1(t)}{n+1} + \frac{\widetilde{D}_2(t) t^2}{(n+1)^2} \right| < \frac{c_N}{\sqrt{n+1}} < 1$$

for $t \in I_n$ and all $n \geq n_N$, where c_N, n_N are constants dependent only on N. Thus, it follows from (4.19)–(4.21) with $F(y) = 1/(1+y)$ that the reciprocal of (4.26) has an expansion

$$\frac{1}{2\rho} \frac{1}{1-e^{-2t/\rho}} \left(1 + \sum_{p=1}^{N-1} \frac{E_p(t)}{(n+1)^p} + \frac{\widetilde{E}_N(t)}{(n+1)^N} \right), \tag{4.27}$$

for all $N \geq 2$, where $E_1(t) = -D_1(t)$ and more generally

$$E_p(t) = (-1)^p D_1^p(t) + \mathcal{O}\left(t^2\right) = \alpha^p \frac{1-(-1)^{n+1}}{2} + \mathcal{O}\left(t^2\right) \tag{4.28}$$

as $t \to 0$ with $|E_p(t)|$ bounded above by a polynomial of degree p independent of n, N, while $|\widetilde{E}_N(t)|$ is bounded above on I_n by a polynomial of degree N whose coefficients depend on N but not on n. Furthermore, observe that

$$\begin{aligned} &- h(x)\epsilon^{n+1}(x) \left(\frac{n+1}{\alpha} \frac{(1-x)^2}{x} r(x) + 2R(x)\left(1-\epsilon^{n+1}(x)\right) \right) = \\ &\qquad - (n+1)(1+x)(1-x)^2 \epsilon^{n+1}(x) \left(-\frac{1}{x} - \frac{2\alpha}{n+1} \frac{R(x)}{r(x)} \frac{1-\epsilon^{n+1}(x)}{(1-x)^2} \right). \end{aligned}$$

It follows from an argument similar to the one given in the first part of the lemma that the above expression has an expansion of the form

$$-(1-x)^2(-1)^{n+1}te^{-t/\rho}\left(1+\sum_{p=1}^{N-1}\frac{G_p(t)t^{p-1}}{(n+1)^p}+\frac{\widetilde{G}_N(t)t^{N-1}}{(n+1)^N}\right), \tag{4.29}$$

for any $N\geq 3$, where

$$G_1(t)=-\alpha\frac{1-(-1)^{n+1}}{2}+\left(1-(-1)^{n+1}\frac{\alpha}{2\rho}\right)t+\mathcal{O}\left(t^2\right) \tag{4.30}$$

and

$$G_2(t)=-\alpha\frac{1-(-1)^{n+1}}{2}\left(1+\frac{\alpha}{2\rho}\right)+\mathcal{O}(t) \tag{4.31}$$

as $t\to 0$, $|G_p(t)|$ is bounded above by a polynomial of degree $p+1$ independent of n,N, while $|\widetilde{G}_N(t)|$ is bounded above on I_n by a polynomial of degree $N+1$ whose coefficients depend on N but not on n. We now get from (4.10), (4.27), and (4.29), that (4.24) and (4.25) do hold for $N\geq 3$ and functions $H_p(t)$ and $\widetilde{H}_N(t)$ that can be computed via (4.17)–(4.18) and whose moduli satisfy the described bounds. The vanishing of $H_p(t)$ as $t\to 0$ can be verified by using (4.17), (4.28), (4.30), and (4.31). To see that $\widetilde{H}_N(t)=\mathcal{O}(t^2)$, observe that

$$h_{n+1}(x)=-(-1)^{n+1}-(-1)^{n+1}(1-t/(n+1))\widetilde{H}_N(t)(n+1)^{-N}+\mathcal{O}\left(t^2\right)$$

by what precedes. Thus, we need to show that $h_{n+1}(x)+(-1)^{n+1}$ is divisible by $(1+x)^2$ (of course, if this were not true, formula (4.6) would not have made sense). Since $h_{n+1}(-1)=-(-1)^{n+1}$, it must hold that $h'_{n+1}(-1)=0$. As was mentioned before (4.6), $h_{n+1}(x)=h_{n+1}(1/x)$ and therefore $x^2h'_{n+1}(x)=-h'_{n+1}(1/x)$, which yields the desired claim. Finally, since $\widetilde{H}_2(t)=H_2(t)+\widetilde{H}_3(t)(n+1)^{-1}$, we can take $N=2$ in (4.25) as well. $\square$

Lemma 4.3.3. *let $x=-1+t/(n+1)$, $t\in I_n$. It holds that*

$$\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x}=\frac{\rho f(t/\rho)}{r(x)}\left(1+\gamma(t/\rho)\sum_{p=1}^{N-1}\frac{K_p(t)}{(n+1)^p}+\gamma(t/\rho)\frac{\widetilde{K}_N(t)}{(n+1)^N}\right) \tag{4.32}$$

for any $N \geq 2$, where $|K_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n, N while $|\widetilde{K}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n.

Proof. Observe that $1 - h^2(x) = \rho^2(1-x)^2 r^{-2}(x)$. Then it follows from (4.24) that

$$\frac{1-h_{n+1}^2(x)}{1-h^2(x)} = 1 + (-1)^{n+1}\gamma(t/\rho)\Big(1+\Gamma_{n+1}(t)\Big)h(x)\frac{r^2(x)}{2\rho^2} - \gamma(t/\rho)^2\Big(1+\Gamma_{n+1}(t)\Big)^2\frac{(1-x)^2}{4}\frac{r^2(x)}{4\rho^2}.$$

Since $h(x)r(x) = -\alpha(1+x)$, expansions (4.22), (4.25) and formulae (4.17)–(4.18) yield that

$$(-1)^{n+1}\Big(1+\Gamma_{n+1}(t)\Big)h(x)\frac{r^2(x)}{2\rho^2} = \sum_{p=1}^{N-1}\frac{H_p^*(t)}{(n+1)^p} + \frac{\widetilde{H}_N^*(t)}{(n+1)^N},$$

for any $N \geq 2$, where $H_1^*(t) = -(-1)^{n+1}(\alpha/\rho)t$, $H_p^*(t) = \mathcal{O}(t^2)$, $p \geq 2$, and $\widetilde{H}_N^*(t) = \mathcal{O}(t^2)$ as $t \to 0$, while $|H_p^*(t)|$ and $|\widetilde{H}_N^*(t)|$ have similar bounds to $|H_p(t)|$ and $|\widetilde{H}_N(t)|$. Furthermore, it clearly holds that

$$\frac{(1-x)^2}{4} = 1 - \frac{t}{n+1} + \frac{1}{4}\frac{t^2}{(n+1)^2} \quad \text{and} \quad \frac{r^2(x)}{4\rho^2} = 1 - \frac{t}{n+1} + \frac{1}{4\rho^2}\frac{t^2}{(n+1)^2}.$$

Therefore, we again get from (4.17)–(4.18) that

$$\Big(1+\Gamma_{n+1}(t)\Big)^2\frac{(1-x)^2}{4}\frac{r^2(x)}{4\rho^2} = 1 + \sum_{p=1}^{N-1}\frac{H_p^{**}(t)}{(n+1)^p} + \frac{\widetilde{H}_N^{**}(t)}{(n+1)^N},$$

for any $N \geq 2$, where $H_1^{**}(t) = -(-1)^{n+1}(\alpha/\rho)t + \mathcal{O}(t^2)$, $H_p^{**}(t) = \mathcal{O}(t^2)$, $p \geq 2$, and $\widetilde{H}_N^{**}(t) = \mathcal{O}(t^2)$ as $t \to 0$ while $|H_p^{**}(t)|$ and $|\widetilde{H}_N^{**}(t)|$ have similar bounds to $|H_p(t)|$ and $|\widetilde{H}_N(t)|$. Altogether, it holds that

$$\frac{1-h_{n+1}^2(x)}{1-h^2(x)} = f^2(t/\rho)\left(1 + \gamma(t/\rho)\sum_{p=1}^{N-1}\frac{J_p(t)}{(n+1)^p} + \gamma(t/\rho)\frac{\widetilde{J}_N(t)}{(n+1)^N}\right),$$

where $J_p(t) = f^{-2}(t/\rho)\Big(H_p^*(t) - \gamma(t/\rho)H_p^{**}(t)\Big)$ and a similar formula holds for $\widetilde{J}_N(t)$. Observe that $f^2(s)$ is a positive function for $s > 0$ that tends to 1 as $s \to \infty$ and such that $f^2(s) =$

$s^2/3 + \mathcal{O}(s^4)$ as $s \to 0$. Therefore, it follows from the corresponding properties of $H_p^*(t)$, $H_p^{**}(t)$, $\widetilde{H}_N^*(t)$, and $\widetilde{H}_N^{**}(t)$ that $J_p(t)$ and $\widetilde{J}_N(t)$ have finite value at the origin and have moduli that satisfy similar bounds to $|H_p(t)|$ and $|\widetilde{H}_N(t)|$. Observe also that there exist n_N and $c_N < 1$ such that

$$\left| \gamma(t/\rho) \sum_{p=1}^{N-1} \frac{J_p(t)}{(n+1)^p} + \gamma(t/\rho) \frac{\widetilde{J}_N(t)}{(n+1)^N} \right| < c_N$$

for all $n \geq n_N$. Therefore, the claim of the lemma now follows from (4.19)–(4.21) applied with $F(y) = \sqrt{1+y}$. $\square$

Lemma 4.3.4. *Let $x = -1 + t/(n+1)$, $t \in I_n$. There exist constants O_p, $p \geq 1$, such that*

$$\frac{2\rho}{\pi} \int_0^{\sqrt{n+1}} \frac{f(t/\rho)}{tr(x)} \mathrm{d}t = \frac{1}{2\pi} \log(n+1) + \frac{A_0}{2} - \frac{1}{\pi} \log(2\rho) + \frac{1}{\pi} \mathcal{L}\left(-1 + \frac{1}{\sqrt{n+1}}\right) + \sum_{p=1}^{N-1} \frac{O_p}{(n+1)^p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

for any $N \geq 1$, where $\mathcal{O}_N(\cdot)$ does not depend on n and

$$\mathcal{L}(x) := \log\left(\frac{4\rho}{\rho(1-x) + r(x)}\right).$$

Proof. Similarly to (4.22), there exist constants r_p^* such that

$$\frac{2\rho}{r(x)} = 1 + \sum_{p=1}^{N-1} \frac{r_p^* t^p}{(n+1)^p} + \frac{\tilde{r}_N^*(t) t^N}{(n+1)^N}, \tag{4.33}$$

for any $N \geq 1$, where $|\tilde{r}_N^*(t)|$ is bounded above on I_n by a constant that depends only on N. Then

$$\mathcal{I}_1 := \frac{2\rho}{\pi} \int_0^{\rho} \frac{f(t/\rho)}{tr(x)} \mathrm{d}t = \frac{1}{\pi} \int_0^1 \frac{f(t)}{t} \mathrm{d}t + \sum_{p=1}^{N-1} \frac{L_p}{(n+1)^p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

where $L_p := (r_p^* \rho^p / \pi) \int_0^1 f(t) t^{p-1} \mathrm{d}t$ and $\mathcal{O}_N(\cdot)$ does not depend on n. Furthermore, it holds that

$$\mathcal{I}_2 := \frac{2\rho}{\pi} \int_{\rho}^{\sqrt{n+1}} \frac{\mathrm{d}t}{tr(x)} = \frac{2\rho}{\pi} \int_{-1+\rho/(n+1)}^{-1+1/\sqrt{n+1}} \frac{dx}{(1+x)r(x)}. \tag{4.34}$$

It can be easily verified by differentiation that an antiderivative of $2\rho/((1+x)r(x))$ is $\log(1+x)+\mathcal{L}(x)$. Again, similarly to (4.22), there exist constants l_p such that

$$\mathcal{L}(x)=\sum_{p=1}^{N-1}\frac{l_p t^p}{(n+1)^p}+\frac{\tilde{l}_N(t)t^N}{(n+1)^N},$$

for any $N\geq 1$, where $|\tilde{l}_N(t)|$ is bounded above on I_n by a constant that depends only on N. Therefore, it holds that

$$\mathcal{I}_2=\frac{1}{2\pi}\log(n+1)-\frac{1}{\pi}\log\rho+\frac{1}{\pi}\mathcal{L}\left(-1+\frac{1}{\sqrt{n+1}}\right)-\sum_{p=1}^{N-1}\frac{l_p\rho^p/\pi}{(n+1)^p}+\mathcal{O}_N\left((n+1)^{-N}\right),$$

where, again, $\mathcal{O}_N(\cdot)$ does not depend on n. Next, we have from (4.33) that

$$\begin{aligned}\mathcal{I}_3:=\frac{2\rho}{\pi}\int_\rho^{\sqrt{n+1}}\frac{f(t/\rho)-1}{tr(x)}\mathrm{d}t=\frac{1}{\pi}\int_1^{\sqrt{n+1}/\rho}\frac{f(t)-1}{t}\mathrm{d}t+\\ \sum_{p=1}^{N-1}\frac{r_p^*\rho^p/\pi}{(n+1)^p}\int_1^{\sqrt{n+1}/\rho}(f(t)-1)t^{p-1}\mathrm{d}t+\frac{\rho^N/\pi}{(n+1)^N}\int_1^{\sqrt{n+1}/\rho}(f(t)-1)\tilde{r}_N^*(\rho t)t^{N-1}\mathrm{d}t\end{aligned}$$

for any $N\geq 1$. Notice that

$$0<1-f(t)<t^2\mathrm{csch}^2(t)<8t^2e^{-2t},\quad t\geq 1. \tag{4.35}$$

Therefore, it holds that

$$0<\int_{\sqrt{n+1}/\rho}^\infty(1-f(t))t^{p-1}\mathrm{d}t\leq C_p(n+1)^{(p+1)/2}e^{-(2/\rho)\sqrt{n+1}}=o_N\left((n+1)^{-N}\right) \tag{4.36}$$

for any $p\geq 0$ and $N\geq 1$ and some constant C_p that depends only on p, where $o_N(\cdot)$ does not depend on n. Moreover, since $|\tilde{r}_N^*(t)|$ is bounded above on I_n by a constant that depends only on N, we have that

$$\left|\int_1^{\sqrt{n+1}/\rho}(f(t)-1)\tilde{r}_N^*(\rho t)t^{N-1}\mathrm{d}t\right|\leq C_N^*\int_1^\infty(1-f(t))t^{N-1}\mathrm{d}t=C_N^{**}.$$

Thus, we can conclude that

$$\mathcal{I}_3 = \frac{1}{\pi}\int_1^\infty \frac{f(t)-1}{t}\mathrm{d}t + \sum_{p=1}^{N-1}\frac{M_p}{(n+1)^p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

where $M_p := (r_p^*\rho^p/\pi)\int_1^\infty (f(t)-1)t^{p-1}\mathrm{d}t$ and $\mathcal{O}_N(\cdot)$ does not depend on n. Since the integral in the statement of the lemma is equal to $\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3$, the desired claim now follows from the definition of A_0 in (1.9), where $O_p = L_p - l_p\rho^p/\pi + M_p$. □

Lemma 4.3.5. *There exist constants T_p such that*

$$\frac{2}{\pi}\int_{-1}^{-1+1/\sqrt{n+1}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x = \frac{1}{2\pi}\log(n+1) + \frac{A_0}{2} - \frac{1}{\pi}\log(2\rho) + \frac{1}{\pi}\mathcal{L}\left(-1+\frac{1}{\sqrt{n+1}}\right) + \sum_{p=1}^{N-1}\frac{T_p}{(n+1)^p} + \mathcal{O}_N\left((n+1)^{-N}\right),$$

for any $N \geq 1$, where $\mathcal{O}_N(\cdot)$ does not depend on n.

Proof. Recall (4.32). It follows from (4.33) and (4.17)–(4.18) that

$$\frac{2\rho}{r(x)}\left(\sum_{p=1}^{N-1}\frac{K_p(t)}{(n+1)^p} + \frac{\widetilde{K}_N(t)}{(n+1)^N}\right) = \sum_{p=1}^{N-1}\frac{S_p(t)}{(n+1)^p} + \frac{\widetilde{S}_N(t)}{(n+1)^N}$$

for any $N \geq 2$, where $|S_p(t)|$ is bounded above by a polynomial of degree $2p$ independent of n, N while $|\widetilde{S}_N(t)|$ is bounded above on I_n by a polynomial of degree $2N$ whose coefficients depend on N but not on n. Similarly to (4.35), it holds that $\gamma(s) < 3se^{-s}$ for $s \geq \log 2$. Because $f(s) \to 1$ as $s \to \infty$, it holds as in (4.36) that

$$0 < \int_{\sqrt{n+1}/\rho}^\infty |\rho S_p(\rho t)|\gamma(t)f(t)\mathrm{d}t \leq C_p(n+1)^{p+1/2}e^{-\sqrt{n+1}/\rho} = o_N\left((n+1)^{-N}\right)$$

for any $p \geq 1$ and $N \geq 1$ and some constant C_p that depends only on p, where $o_N(\cdot)$ does not depend on n. Moreover, a similar estimate takes place if $S_p(t)$ is replaced by $\widetilde{S}_N(t)$. The claim of the lemma now follows by making a substitution $x = -1 + t/(n+1)$ to get

$$\frac{2}{\pi}\int_{-1}^{-1+1/\sqrt{n+1}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x = \frac{2}{\pi}\int_0^{\sqrt{n+1}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x}\frac{\mathrm{d}t}{t}$$

and then using Lemmas 4.3.3 and 4.3.4, where $T_p = O_p + (\rho/\pi)\int_0^\infty f(t)\gamma(t)S_p(\rho t)\mathrm{d}t$ (since $T_1/(n+1) = \mathcal{O}_N((n+1)^{-1})$, the claim indeed holds for all $N \geq 1$). $\square$

Lemma 4.3.6. *It holds that*

$$\frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1-\delta_\alpha^{n+1}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\,\mathrm{d}x = \frac{1}{2\pi}\log(n+1)+ \\ \frac{1}{\pi}\log\left(\frac{4\rho}{|\alpha|}\right) - \frac{1}{\pi}\mathcal{L}\left(-1+\frac{1}{\sqrt{n+1}}\right) + o_N\left((n+1)^{-N}\right)$$

for any integer $N \geq 1$, where $o_N(\cdot)$ is independent of n, but does depend on N.

Proof. Since $|h_{n+1}(x)|, |h(x)| \leq 1$ when $x \in [-1,1]$, it holds that

$$\begin{aligned}\left|\frac{\sqrt{1-h^2(x)} - \sqrt{1-h_{n+1}^2(x)}}{1-x^2}\right| &= \frac{\left|h_{n+1}^2(x) - h^2(x)\right|}{(1-x^2)\left(\sqrt{1-h^2(x)}+\sqrt{1-h_{n+1}^2(x)}\right)} \\ &\leq \frac{2|h_{n+1}(x)-h(x)|}{(1-x^2)\sqrt{1-h^2(x)}} = \frac{2}{\rho}\frac{r(x)}{(1+x)}\frac{|h_{n+1}(x)-h(x)|}{(1-x)^2}.\end{aligned}$$

Since $r(x) \leq 2$, $x \in [-1,1]$, we obtain from (4.5) that

$$\left|\frac{\sqrt{1-h^2(x)} - \sqrt{1-h_{n+1}^2(x)}}{1-x^2}\right| \leq C(n+1)^{3/2}e^{-\sqrt{n+1}/\rho}$$

for $-1+1/\sqrt{n+1} \leq x \leq 1-\delta_\alpha^{n+1}$ and some constant C. Therefore, it holds that

$$\left|\frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1-\delta_\alpha^{n+1}} \frac{\sqrt{1-h^2(x)} - \sqrt{1-h_{n+1}^2(x)}}{1-x^2}\,\mathrm{d}x\right| = o_N\left((n+1)^{-N}\right)$$

for any $N \geq 1$, where $o_N(\cdot)$ is independent of n, but does depend on N. Furthermore, since $r(x) \geq 2|\alpha|\rho$ for $x \in [-1,1]$, it holds that

$$\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1}\frac{\sqrt{1-h^2(x)}}{1-x^2}\mathrm{d}x = \frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1}\frac{\rho\,\mathrm{d}x}{(1+x)r(x)} \leq \frac{\delta_\alpha^{n+1}}{|\alpha|\pi} = o_N\left((n+1)^{-N}\right)$$

for any $N \geq 1$ by the very definition of δ_α, where, again, $o_N(\cdot)$ is independent of n, but does depend on N. The observation made after (4.34) allows us now to conclude that

$$\frac{2}{\pi}\int_{-1+1/\sqrt{n+1}}^{1} \frac{\rho \mathrm{d}x}{(1+x)r(x)} = \frac{1}{2\pi}\log(n+1) + \frac{1}{\pi}\log\left(\frac{4\rho}{|\alpha|}\right) - \frac{1}{\pi}\mathcal{L}\left(-1+\frac{1}{\sqrt{n+1}}\right),$$

which finishes the proof of the lemma. □

Lemma 4.3.7. *When $\alpha > 0$, it holds that*

$$\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x = 1 + o_N\left((n+1)^{-N}\right)$$

for any $N \geq 1$, where $o_N(\cdot)$ is independent of n, but does depend on N.

Proof. It follows from (4.10) and (4.14) that

$$h_{n+1}(x) = h(x) - h(x)\frac{\epsilon^{n+1}(x)X_{n+1}(x)}{(1-x)^2 + \epsilon^{n+1}(x)Y_{n+1}(x)},$$

where

$$X_{n+1}(x) := \frac{R(x)}{\alpha\rho^2}\left((n+1)r(x)\frac{(1-x)^2}{x} + 2\alpha R(x)\big(1-\epsilon^{n+1}(x)\big)\right)$$

and

$$Y_{n+1}(x) := \frac{R(x)}{\rho^2}\left(2\alpha(x+1) - R(x)\epsilon^{n+1}(x)\right).$$

Therefore, we can write

$$1 - h_{n+1}^2(x) = \rho^2\frac{(1-x)^2}{r^2(x)} + h^2(x)\frac{(1-x)^2\epsilon^{n+1}(x)X_{n+1}(x)}{((1-x)^2+\epsilon^{n+1}(x)Y_{n+1}(x))^2}\times \left(2 - \epsilon^{n+1}(x)\frac{X_{n+1}(x)-2Y_{n+1}(x)}{(1-x)^2}\right).$$

We have that

$$\frac{X_{n+1}(x)-2Y_{n+1}(x)}{(1-x)^2} = \frac{R(x)}{\rho^2(1-x)^2}\left(2S(x) + (n+1)\frac{(1-x)^2r(x)}{\alpha x}\right) = 2 + (n+1)\frac{r(x)R(x)}{\alpha\rho^2 x},$$

where we used (4.14) once more. Hence, it holds that

$$\frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}=\frac{\epsilon^{(n+1)/2}(x)V_{n+1}(x)}{(1-x)^2+\epsilon^{n+1}(x)Y_{n+1}(x)}, \tag{4.37}$$

where

$$V_{n+1}(x):=-\frac{2}{\rho}\frac{h(x)R(x)}{1+x}\frac{\sqrt{\left(1-\epsilon^{n+1}(x)\left(1+(n+1)\frac{r(x)R(x)}{2\alpha\rho^2x}\right)\right)\times}}{\times\left(1-\epsilon^{n+1}(x)+(n+1)\frac{(1-x)^2r(x)}{2\alpha xR(x)}\right)+\left(\rho^2\frac{(1-x)^2+\epsilon^{n+1}(x)Y_{n+1}(x)}{2\epsilon^{(n+1)/2}(x)h(x)r(x)R(x)}\right)^2}$$

(observe that $-h(x)>0$). Recall that $\delta_\alpha=\epsilon^{1/3}(1)$. In particular, we get from (4.11) that

$$\frac{(1-x)^2}{\epsilon^{(n+1)/2}(x)}\leq\epsilon^{\frac{n+1}{6}}(1)\left(\frac{\epsilon(1)}{\epsilon(1-\delta_\alpha^{n+1})}\right)^{\frac{n+1}{2}}=(1+o(1))\epsilon^{\frac{n+1}{6}}(1)=o_N\left((n+1)^{-N}\right) \tag{4.38}$$

for $1-\delta_\alpha^{n+1}\leq x\leq 1$. Since

$$Y_{n+1}(x)=(2\alpha/\rho^2)(x+1)R(x)+o_N\left((n+1)^{-N}\right) \tag{4.39}$$

on any fixed small enough neighborhood of 1, it holds that

$$V_{n+1}(x)=-\frac{2}{\rho}\frac{h(x)R(x)}{1+x}+o_N\left((n+1)^{-N}\right)=\frac{4\alpha}{\rho}+o_N\left((n+1)^{-N}\right) \tag{4.40}$$

uniformly for $1-\delta_\alpha^{n+1}\leq x\leq 1$. Let

$$Z_{n+1}(x):=\sqrt{Y_{n+1}(x)}-\frac{x-1}{2}\left((n+1)\frac{1-x}{x}\frac{\sqrt{Y_{n+1}(x)}}{r(x)}+\frac{Y_{n+1}'(x)}{\sqrt{Y_{n+1}(x)}}\right).$$

It follows from the definition of $Z_{n+1}(x)$ and an estimate similar to (4.38) that

$$\begin{aligned}\int_{1-\delta_\alpha^{n+1}}^1\frac{\epsilon^{(n+1)/2}(x)Z_{n+1}(x)\,\mathrm{d}x}{(1-x)^2+\epsilon^{n+1}(x)Y_{n+1}(x)}&=\arctan\left(\frac{x-1}{\sqrt{\epsilon^{n+1}(x)Y_{n+1}(x)}}\right)\Bigg|_{1-\delta_\alpha^{n+1}}^1\\&=\frac{\pi}{2}-\arctan\left(\mathcal{O}(1)\epsilon^{\frac{n+1}{6}}(1)\right)=\frac{\pi}{2}+o_N\left((n+1)^{-N}\right).\end{aligned}$$

Furthermore, we get from (4.39), the definition of $Z_{n+1}(x)$, and (4.40) that

$$Z_{n+1}(x) = \frac{4\alpha}{\rho} + o_N\left((n+1)^{-N}\right) = V_{n+1}(x) + o_N\left((n+1)^{-N}\right)$$

uniformly for $1-\delta_\alpha^{n+1} \leq x \leq 1$. Therefore, (4.37) yields that

$$\begin{aligned}\frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2}\mathrm{d}x &= \frac{2}{\pi}\int_{1-\delta_\alpha^{n+1}}^{1} \frac{\epsilon^{(n+1)/2}(x)\Big(Z_{n+1}(x) + o_N\left((n+1)^{-N}\right)\Big)}{(1-x)^2 + \epsilon^{n+1}(x)Y_{n+1}(x)}\mathrm{d}x \\ &= 1 + o_N\left((n+1)^{-N}\right),\end{aligned}$$

where we used positivity of the integrand for the last estimate. □

Proof of Theorem 4.1.2. The claim follows from formula (4.6) and Lemmas 4.3.5–4.3.7. □

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